VECTOR ANALYSIS

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PREFACE

A sound mathematical training for the modern engineer is a *sine qua non* for attaining new heights in all aspects of engineering practice. One of the areas of mathematics that plays a big role in the mathematical education of the engineer is vector analysis, which is now invariably included in the curriculum of higher mathematics in engineering colleges.

The present collection of problems in vector analysis contains the required minimum of problems and exercises for the course of vector analysis of engineering colleges.

Each section starts with a brief review of theory and detailed solutions of a sufficient number of typical problems. The text contains 100 worked problems and there are 314 problems left to the student. There are also a certain number of problems of an applied nature that have been chosen so that their analysis does not require supplementary information in specialized fields. The material of the sixth chapter is devoted to curvilinear coordinates and the basic operations of vector analysis in curvilinear coordinates. Its purpose is to give the reader at least a few problems to develop the necessary skills.

The exposition in this text follows closely the lines currently employed at the chair of higher mathematics of the Moscow Power Institute.

The present text may be regarded as a short course in vector analysis in which the basic facts are given without proof but with illustrative examples of a practical nature. Hence this problem book may be used in a recapitulation of the essentials of vector analysis or as a text for readers who wish merely to master the techniques of vector analysis, while dispensing with the proofs of propositions and theorems.

In compiling this problem book, the authors made extensive use of material in published courses of vector

calculus and collections of problems. Many problems were made up by the authors themselves.

This collection of problems is designed for students of day and evening departments at engineering colleges and also for correspondence students with a background of vector algebra and calculus as given in the first two years of college study.

We would like to express our sincere gratitude to Professor V. P. Gromov (the Moscow Krupskaya Pedagogical Institute), Professor A. V. Efimov and Associate Professors I. M. Petrov, B. I. Fridlender, and V. N. Zemskov (the Institute of Electronics) for their thorough scrutiny of the manuscript of the book and for valuable tuggestions and remarks that were made full use of in she final editing.

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CHAPTER I

THE VECTOR FUNCTION OF A SCALAR ARGUMENT

Sec. 1. The hodograph of a vector function

Definition 1. A vector $\mathbf{r}$ is said to be the *vector function* of a scalar argument t if each value of the scalar taken from the domain of admissible values is associated with a definite value of the vector $\mathbf{r}$. This can be written as follows:

$$\mathbf{r} = \mathbf{r}\,(t).$$

If the vector $\mathbf{r}$ is a function of the scalar argument t,

$$\mathbf{r} = \mathbf{r}\,(t),$$

then the coordinates x, y, z of the vector $\mathbf{r}$ are also functions of t:

$$x = x\,(t), \quad y = y\,(t), \quad z = z\,(t).$$

Conversely, if the coordinates of the vector $\mathbf{r}$ are functions of t, then the vector $\mathbf{r}$ itself is also a function of t:

$$\mathbf{r} = x\,(t)\,\mathbf{i} + y\,(t)\,\mathbf{j} + z\,(t)\,\mathbf{k}.$$

Thus, specifying a vector function $\mathbf{r}\,(t)$ is the same as specifying three scalar functions $x\,(t)$, $y\,(t)$, $z\,(t)$.

Definition 2. The *hodograph* of the vector function $\mathbf{r}\,(t)$ of a scalar argument is the locus described by the terminus of the vector $\mathbf{r}\,(t)$, as the scalar t varies, when the origin of the vector $\mathbf{r}\,(t)$ is fixed at a point O in space (Fig. 1).

The hodograph of a radius vector $\mathbf{r} = \mathbf{r}\,(t)$ of a moving point is the trajectory L of that point. Some other line L_1

(Fig. 2) is the hodograph of the velocity $\mathbf{v} = \mathbf{v}(t)$ of that point. Thus, if a material point (particle) is in motion around a circle with constant velocity, $|\mathbf{v}| =$ constant, then its hodograph of velocities is likewise a circle with centre at O_1 and with radius equal to $|\mathbf{v}|$.

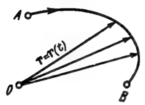

Fig. 1

Example 1. Construct the hodograph of the vector $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$.

Solution. 1°. This construction may be carried out by using points and setting up a table:

t	0	1	2	3	4
$\mathbf{r}$	0	$\mathbf{i}+\mathbf{j}+\mathbf{k}$	$2\mathbf{i}+2\mathbf{j}+4\mathbf{k}$	$3\mathbf{i}+3\mathbf{j}+9\mathbf{k}$	$4\mathbf{i}+4\mathbf{j}+16\mathbf{k}$

2°. Alternative solution. Denote by x, y, z the coordinates of vector $\mathbf{r}$; we have

$$x = t, \quad y = t, \quad z = t^2.$$

Eliminating the parameter t from these equations, we get equations of the surfaces $y = x$, $z = x^2$, the line L

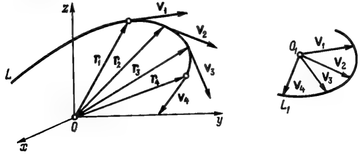

Fig. 2

of intersection of which is what defines the hodograph of the vector $\mathbf{r}(t)$ (Fig. 3).

1. Construct the hodographs of the following vectors:

(a) $\mathbf{r}=2\mathbf{i}+t^2\mathbf{j}-t^2\mathbf{k}$.

(b) $\mathbf{r}=\frac{t^2+1}{(t+1)^2}\,\mathbf{i}+\frac{2t}{(t+1)^2}\,\mathbf{j}$.

(c) $\mathbf{r}=\cos t\cdot\mathbf{i}+\sin t\cdot\mathbf{j}+\mathbf{k}$.

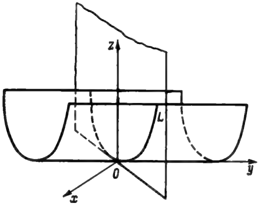

Fig. 3

(d) $\mathbf{r}=t\mathbf{i}+\frac{1}{3}t^2\mathbf{j}+\frac{1}{9}t^3\mathbf{k}$.

(e) $\mathbf{r}=\frac{2t\mathbf{i}+2t\mathbf{j}+(t^2-2)\,\mathbf{k}}{t^2+2}$.

Sec. 2. The limit and continuity of a vector function of a scalar argument

Suppose a vector function $\mathbf{r}=\mathbf{r}(t)$ of a scalar argument t is defined in some neighbourhood of the value t_0 of the argument t, except perhaps for the value t_0 itself.

Definition 1. A constant vector $\mathbf{A}$ is said to be the *limit* of the vector $\mathbf{r}(t)$, as $t\to t_0$, if for any $\varepsilon>0$ there is a $\delta>0$ such that for all $t\neq t_0$ that satisfy the condition $|t-t_0|<\delta$ the following inequality holds true:

$$|\mathbf{r}(t)-\mathbf{A}|<\varepsilon.$$

As in the case of ordinary calculus, we write $\lim\limits_{t \to t_0} \mathbf{r}(t) = \mathbf{A}$.

Geometrically (Fig. 4), this means that the vector $\mathbf{r}(t)$ tends, as $t \to t_0$, to the vector $\mathbf{A}$ both in length and in direction.

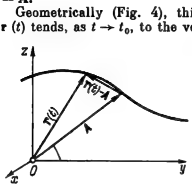

Fig. 4

Definition 2. A vector $\alpha(t)$ is said to be *infinitesimal*, as $t \to t_0$, if $\alpha(t)$ has a limit, as $t \to t_0$, and that limit is zero:

$$\lim_{t \to t_0} \alpha(t) = 0,$$

or, what is the same thing, if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $t \neq t_0$ that satisfy the condition $|t - t_0| < \delta$, the inequality $|\alpha(t)| < \varepsilon$ holds true.

Example 1. Show that the vector $\alpha(t) = t\mathbf{i} + \sin t\mathbf{j}$ is infinitesimal when $t \to 0$.

Solution. We have

$$|\alpha(t)| = |t\mathbf{i} + \sin t\mathbf{j}| \leqslant |t| + |\sin t| \leqslant 2|t|.$$

From this it is evident that if for every $\varepsilon > 0$ we take $\delta = \varepsilon/2$, then for $|t - 0| < \delta = \varepsilon/2$ we have $|\alpha(t)| < \varepsilon$. By the definition, this means that $\alpha(t)$ is an infinitesimal vector when $t \to 0$.

2. Show that the limit of the modulus of a vector is equal to the modulus of its limit (if the limit exists).
3. Demonstrate that for a vector function $\mathbf{r}(t)$ to have a limit $\mathbf{A}$, as $t \to t_0$, it is necessary and sufficient that $\mathbf{r}(t)$ be representable in the form

$$\mathbf{r}(t) = \mathbf{A} + \alpha(t),$$

where $\alpha(t)$ is an infinitesimal vector when $t \to t_0$.
4. Show that if the vector functions $\mathbf{a}(t)$ and $\mathbf{b}(t)$ have limits as $t \to t_0$,

$$\lim_{t \to t_0} \mathbf{a}(t) = \mathbf{A}, \quad \lim_{t \to t_0} \mathbf{b}(t) = \mathbf{B},$$

then their sum $\mathbf{a}(t) + \mathbf{b}(t)$ and their difference $\mathbf{a}(t) - \mathbf{b}(t)$ also have limits as $t \to t_0$, and

$$\lim_{t \to t_0} [\mathbf{a}(t) \pm \mathbf{b}(t)] = \mathbf{A} \pm \mathbf{B}.$$

5. Let

$$\lim_{t \to t_0} \mathbf{a}(t) = \mathbf{A}, \quad \lim_{t \to t_0} \mathbf{b}(t) = \mathbf{B}.$$

Prove that

$$\lim_{t \to t_0} (\mathbf{a}(t), \mathbf{b}(t)) = (\mathbf{A}, \mathbf{B}),$$

where $(\mathbf{a}(t), \mathbf{b}(t))$ is a scalar product of the vector functions $\mathbf{a}(t)$ and $\mathbf{b}(t)$.

6. Let

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \quad \mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

Show that if $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{A}$, then

$$\lim_{t \to t_0} x(t) = a_1, \quad \lim_{t \to t_0} y(t) = a_2, \quad \lim_{t \to t_0} z(t) = a_3.$$

Find the following limits:

7. $\lim\limits_{t \to 0} \left(\frac{\sin t}{t}\,\mathbf{i} + \frac{\cos t - 1}{2t}\,\mathbf{j} + e^t\,\mathbf{k} \right)$.

8. $\lim\limits_{t \to 0} \left(\frac{1-\sqrt{t}}{1-t}\,\mathbf{i} + \frac{t}{1+t}\,\mathbf{j} + \mathbf{k} \right)$.

9. $\lim\limits_{t \to \pi} \left(\frac{\sin t}{t}\,\mathbf{i} + \cos t \cdot \mathbf{j} + \frac{\mathbf{k}}{t+\pi} \right)$.

10. $\lim\limits_{t \to \pi} \left(\frac{\sin t}{t-\pi}\,\mathbf{i} + \frac{1+\cos t}{t}\,\mathbf{j} + \frac{t}{\pi}\,\mathbf{k} \right)$.

11. $\lim\limits_{t \to 1} \left(\frac{e^t - e}{t-1}\,\mathbf{i} + \frac{\ln t}{1-t}\,\mathbf{j} + 2\mathbf{k} \right)$.

Definition 3. A vector function $\mathbf{r} = \mathbf{r}(t)$ defined in some neighbourhood of the value $t = t_0$ is said to be *continuous* when $t = t_0$ if

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0).$$

In other words, $\mathbf{r} = \mathbf{r}(t)$ is continuous for $t = t_0$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all t that satisfy the condition $|t - t_0| < \delta$ the inequality $|\mathbf{r}(t) - \mathbf{r}(t_0)| < \varepsilon$ holds true.

The hodograph of a continuous vector function of a scalar argument is a continuous curve.

12. Start with the familiar inequality $|\mathbf{a} - \mathbf{b}| \geqslant ||\mathbf{a}| - |\mathbf{b}||$ and demonstrate that the continuity of a vector function implies the continuity of its modulus. Is the converse true?

13. Show that if $\mathbf{a}(t)$ and $\mathbf{b}(t)$ are continuous for $t = t_0$, then the vector function $\mathbf{a}(t) \pm \mathbf{b}(t)$ is also continuous for $t = t_0$.

14. A vector function $\mathbf{a}(t) + \mathbf{b}(t)$ is continuous for $t = t_0$. Does it follow from this that the vectors $\mathbf{a}(t)$ and $\mathbf{b}(t)$ are also continuous when $t = t_0$?

15. Prove that if $\mathbf{a}(t)$ and $\mathbf{b}(t)$ are continuous vector functions, then their scalar product $(\mathbf{a}(t), \mathbf{b}(t))$ and vector product $[\mathbf{a}(t), \mathbf{b}(t)]$ are also continuous.

Sec. 3. The derivative of a vector function with respect to a scalar argument

Suppose a vector function $\mathbf{r} = \mathbf{r}(t)$ is defined for all t on the interval (t_0, t_1). Take some value $t \in (t_0, t_1)$, then give t an increment Δt such that $t + \Delta t \in (t_0, t_1)$ and find the corresponding increment $\Delta \mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$ in the vector function $\mathbf{r}(t)$. Now consider the ratio $\Delta \mathbf{r}/\Delta t$.

Definition. If, as $\Delta t \to 0$, the ratio $\Delta \mathbf{r}/\Delta t$ has a limit, then that limit is called the *derivative* of the vector function $\mathbf{r} = \mathbf{r}(t)$ with respect to the scalar argument t for a given value t of the argument and is denoted as $d\mathbf{r}(t)/dt$ or $\mathbf{r}'(t)$ or $\dot{\mathbf{r}}(t)$. Thus,

$$\frac{d\mathbf{r}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}.$$

In this case the vector function $\mathbf{r} = \mathbf{r}(t)$ is said to be *differentiable*.

16. Show that if the vector function $\mathbf{r} = \mathbf{r}(t)$ has a derivative for some value t of the argument, then it is continuous for that value t.

The derivative of a vector function $\mathbf{r}(t)$ of a scalar argument t is a vector directed along the tangent to the

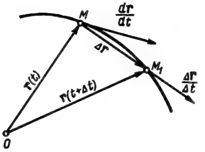

Fig. 5

hodograph of the original vector at the point under consideration (Fig. 5). The vector $d\mathbf{r}/dt$ is in the direction of the terminus of the vector $\mathbf{r}(t)$ as it moves along the hodograph when the parameter t increases.

Suppose $\mathbf{r} = \mathbf{r}(t)$ is the radius vector of a moving point. Then the vector $\mathbf{v} = d\mathbf{r}/dt$ is the velocity vector of that point.

Suppose

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k},$$

where the functions $x(t)$, $y(t)$, $z(t)$ are differentiable at the point t. Then there exists $d\mathbf{r}/dt$ for that value of t and

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}. \quad (1)$$

Example 1. Find $d\mathbf{r}/dt$ if $\mathbf{r} = \mathbf{i}a\cos t + \mathbf{j}b\sin t$ (the point is moving in an ellipse).

Solution. From formula (1),

$$\frac{d\mathbf{r}}{dt} = -\mathbf{i}a\sin t + \mathbf{j}b\cos t.$$

By analogy with the differential of a scalar function, the *differential of a vector function* $\mathbf{r} = \mathbf{r}(t)$ is a vector $d\mathbf{r}$ defined by the equality

$$d\mathbf{r} = \frac{d\mathbf{r}}{dt} \cdot dt,$$

where $dt = \Delta t$ is the increment in the scalar argument t.

As in the case of scalar functions,

$$\Delta \mathbf{r} = d\mathbf{r} + \boldsymbol{\alpha} \cdot \Delta t,$$

where $\boldsymbol{\alpha} = \boldsymbol{\alpha}(t, \Delta t) \to 0$ as $\Delta t \to 0$.

Basic rules for differentiating a vector function

Assume that all functions being considered (both scalar and vector) are continuous and differentiable.

1°. If $\mathbf{c}$ is a constant vector, then $d\mathbf{c}/dt = 0$.

2°. The derivative of a sum of vector functions is equal to the sum of the derivatives of the summands:

$$\frac{d(\mathbf{a}(t) + \mathbf{b}(t))}{dt} = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}$$

3°. Suppose a vector function $\mathbf{a}(t)$ is multiplied by a scalar function $m(t)$ of the same scalar argument. Then

$$\frac{d(m\mathbf{a})}{dt} = m\frac{d\mathbf{a}}{dt} + \frac{dm}{dt}\mathbf{a}$$

4°. $\dfrac{d(\mathbf{a}, \mathbf{b})}{dt} = \left(\mathbf{a}, \dfrac{d\mathbf{b}}{dt}\right) + \left(\dfrac{d\mathbf{a}}{dt}, \mathbf{b}\right)$

5°. $\dfrac{d[\mathbf{a}, \mathbf{b}]}{dt} = \left[\dfrac{d\mathbf{a}}{dt}, \mathbf{b}\right] + \left[\mathbf{a}, \dfrac{d\mathbf{b}}{dt}\right]$

(In this formula, the order of the factors $\mathbf{a}$ and $\mathbf{b}$ in the right-hand member must be the same as that in the left-hand member.)

Let us prove formula 4°. We set $\varphi(t) = (\mathbf{a}(t), \mathbf{b}(t))$. Give t an increment Δt; then, by the distributive property, we have for the scalar product

$$\Delta\varphi = \varphi(t + \Delta t) - \varphi(t) = (\mathbf{a} + \Delta\mathbf{a}, \mathbf{b} + \Delta\mathbf{b}) - (\mathbf{a}, \mathbf{b})$$
$$= (\Delta\mathbf{a}, \mathbf{b}) + (\mathbf{a}, \Delta\mathbf{b}) + (\Delta\mathbf{a}, \Delta\mathbf{b}),$$

whence

$$\frac{\Delta\varphi}{\Delta t} = \left(\frac{\Delta\mathbf{a}}{\Delta t}, \mathbf{b}\right) + \left(\mathbf{a}, \frac{\Delta\mathbf{b}}{\Delta t}\right) + \left(\frac{\Delta\mathbf{a}}{\Delta t}, \Delta\mathbf{b}\right). \qquad (2)$$

It is given that the functions $\mathbf{a}(t)$ and $\mathbf{b}(t)$ have derivatives for the value t of the argument and, hence, are continuous for that value of t. Therefore

$$\lim_{\Delta t\to 0}\frac{\Delta \mathbf{a}}{\Delta t}=\frac{d\mathbf{a}}{dt}, \quad \lim_{\Delta t\to 0}\frac{\Delta \mathbf{b}}{\Delta t}=\frac{d\mathbf{b}}{dt}, \text{ and } \lim_{\Delta t\to 0}\Delta \mathbf{b}=0.$$

Passing to the limit in (2) as $\Delta t\to 0$, we obtain

$$\frac{d(\mathbf{a},\mathbf{b})}{dt}=\left(\frac{d\mathbf{a}}{dt},\mathbf{b}\right)+\left(\mathbf{a},\frac{d\mathbf{b}}{dt}\right).$$

17. Given $\mathbf{r}=\mathbf{r}(t)$. Find the derivatives:

(a) $\frac{d}{dt}(\mathbf{r}^2)$, (b) $\frac{d}{dt}\left(\mathbf{r},\frac{d\mathbf{r}}{dt}\right)$, (c) $\frac{d}{dt}\left[\mathbf{r},\frac{d\mathbf{r}}{dt}\right]$.

18. Prove that if the modulus $|\mathbf{r}|$ of the vector function $\mathbf{r}=\mathbf{r}(t)$ remains constant for all values of t, then $d\mathbf{r}/dt\perp\mathbf{r}$. What is the geometrical meaning of this fact?

19. Prove that if $\mathbf{e}$ is a unit vector in the direction of the vector $\mathbf{E}$, then

$$[\mathbf{e},d\mathbf{e}]=\frac{[\mathbf{E},d\mathbf{E}]}{|\mathbf{E}|^2}.$$

20. Suppose

$$\mathbf{u}=u_1(x,y,z,t)\,\mathbf{i}+u_2(x,y,z,t)\,\mathbf{j}+u_3(x,y,z,t)\,\mathbf{k},$$

where u_1, u_2, u_3 are continuously differentiable functions of their arguments, and x, y, z are continuously differentiable functions of t. Show that

$$\frac{d\mathbf{u}}{dt}=\frac{\partial\mathbf{u}}{\partial t}+\frac{\partial\mathbf{u}}{\partial x}\frac{dx}{dt}+\frac{\partial\mathbf{u}}{\partial y}\frac{dy}{dt}+\frac{\partial\mathbf{u}}{\partial z}\frac{dz}{dt}.$$

21. Find the trajectory of motion for which the radius vector $\mathbf{r}(t)$ of a moving point satisfies the condition $d\mathbf{r}/dt=[\mathbf{a},\mathbf{r}]$, where $\mathbf{a}$ is a constant vector.

The derivative $d\mathbf{r}/dt$ of the vector function $\mathbf{r}(t)$ of a scalar argument is a vector function of the same argument. If there exists a derivative of $d\mathbf{r}/dt$, then it is called the *second derivative* and is denoted $d^2\mathbf{r}/dt^2$. Generally,

$$\frac{d^n\mathbf{r}}{dt^n}=\frac{d}{dt}\left(\frac{d^{n-1}\mathbf{r}}{dt^{n-1}}\right),\quad n=1,2,\ldots.$$

22. Given the radius vector of a point moving in space:

$$\mathbf{r}\{a\sin t,\ -a\cos t,\ bt^2\}$$

(t is time, and a and b are constants). Find the hodographs of velocity and acceleration.

23. Given: $\mathbf{r} = \mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t$, where ω, $\mathbf{a}$, $\mathbf{b}$ are constants. Prove that

(1) $\left[\mathbf{r}, \frac{d\mathbf{r}}{dt}\right] = [\omega\mathbf{a}, \mathbf{b}]$,

(2) $\frac{d^2\mathbf{r}}{dt^2} + \omega^2\mathbf{r} = 0$.

24. Show that if $\mathbf{r} = \mathbf{a}e^{\omega t} + \mathbf{b}e^{-\omega t}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, then $d^2\mathbf{r}/dt - \omega^2\mathbf{r} = 0$.

25. Show that the modulus of the differential of the radius vector of a point is equal to the differential of the length of the arc described by the point.

26. Suppose $\mathbf{a} = \mathbf{a}(u)$ is a vector function of a scalar u, where u in turn is a certain scalar function of the basic scalar t. Assuming $\mathbf{a}(u)$ and $u = u(t)$ to be differentiable the necessary number of times, find an expression for the derivatives of the composite function $d\mathbf{a}/dt$, $d^2\mathbf{a}/dt^2$.

Sec. 4. Integrating a vector function of a scalar argument

Definition 1. We will say that the vector function $\mathbf{A}(t)$ is the *primitive* of the vector function $\mathbf{a}(t)$ when $t_0 < t < t_1$ if $\mathbf{A}(t)$ is differentiable and

$$\frac{d\mathbf{A}}{dt} = \mathbf{a}(t), \quad t \in (t_0, t_1).$$

Definition 2. The collection of all primitive functions of $\mathbf{a}(t)$ is termed the *indefinite integral* of the vector function of a scalar argument $\mathbf{a} = \mathbf{a}(t)$. As in integral calculus, the indefinite integral of a vector function is denoted by the symbol $\int$, and we have

$$\int \mathbf{a}(t)\,dt = \mathbf{A}(t) + \mathbf{C},$$

where $\mathbf{A}(t)$ is one of the primitive functions of $\mathbf{a}(t)$, and $\mathbf{C}$ is an arbitrary constant vector.

The following properties hold true for integrals of vector functions:

1°. $\int \alpha\mathbf{a}(t)\,dt = \alpha\int \mathbf{a}(t)\,dt$ (α is a numerical constant).

2°. $\int (\mathbf{a}(t) \pm \mathbf{b}(t))\,dt = \int \mathbf{a}(t)\,dt \pm \int \mathbf{b}(t)\,dt.$

27. Show that if $\mathbf{c}$ is a constant vector, and $\mathbf{a}(t)$ is a variable vector, then

$$\int (\mathbf{c}, \mathbf{a}(t))\,dt = \left(\mathbf{c}, \int \mathbf{a}(t)\,dt\right),$$

$$\int [\mathbf{c}, \mathbf{a}(t)]\,dt = \left[\mathbf{c}, \int \mathbf{a}(t)\,dt\right].$$

If

$$\mathbf{a}(t) = a_1(t)\,\mathbf{i} + a_2(t)\,\mathbf{j} + a_3(t)\,\mathbf{k},$$

then

$$\int \mathbf{a}(t)\,dt = \mathbf{i}\int a_1(t)\,dt + \mathbf{j}\int a_2(t)\,dt + \mathbf{k}\int a_3(t)\,dt. \quad (1)$$

That is, the integration of a vector function reduces to three ordinary integrations.

Example 1. Find the indefinite integral for the vector function $\mathbf{a}(t) = \mathbf{i}\cos t + \mathbf{j}e^{-t} + \mathbf{k}$.

Solution. According to formula (1),

$$\int \mathbf{a}(t)\,dt = \mathbf{i}\int \cos t\,dt + \mathbf{j}\int e^{-t}\,dt + \mathbf{k}\int dt =$$
$$= \mathbf{i}\sin t - \mathbf{j}e^{-t} + \mathbf{k}t + \mathbf{c},$$

where $\mathbf{c}$ is an arbitrary constant vector.

Find the integrals of the following vector functions:

28. $\mathbf{a}(t) = te^t\mathbf{i} + \sin^2 t\cdot\mathbf{j} - \frac{\mathbf{k}}{1+t^2}$.

29. $\mathbf{a}(t) = \frac{t\mathbf{i}}{1+t^2} + te^{t^2}\mathbf{j} + \cos t\cdot\mathbf{k}$.

30. $\mathbf{a}(t) = \cos t e^{\sin t}\cdot\mathbf{i} - t\cos t^2\cdot\mathbf{j} + \mathbf{k}$.

31. $\mathbf{a}(t) = \frac{1}{2}t^2\mathbf{i} - t\sin t\cdot\mathbf{j} + 2^t\mathbf{k}$.

Let a vector function $\mathbf{a}(t)$ be defined and continuous over a certain interval $[t_0, T]$, which is the range of the argument t.

Definition 3. We define the *definite integral* of a vector function $\mathbf{a}(t)$ on the interval $[t_0, T]$ as the limit of the

vector integral sums

$$\sigma = \sum_{k=0}^{n-1} \mathbf{a}(\tau_k)\,\Delta t_k, \qquad \tau_k \in [t_k, t_{k+1}]$$

as the length Δt of the largest of the subintervals $[t_k, t_{k+1}]$ $(k = 0, 1, \ldots, n-1)$ into which the interval $[t_0, T]$ is partitioned tends to zero:

$$\int_{t_0}^{T} \mathbf{a}(t)\,dt = \lim_{\Delta t \to 0} \sum_{k=0}^{n-1} \mathbf{a}(\tau_k)\,\Delta t_k.$$

The following formula holds true:

$$\int_{t_0}^{T} \mathbf{a}(t)\,dt = \mathbf{A}(T) - \mathbf{A}(t_0), \tag{2}$$

where $\mathbf{A}(t)$ is some primitive for the function $\mathbf{a}(t)$ on the interval $[t_0, T]$.

If

$$\mathbf{a}(t) = a_1(t)\,\mathbf{i} + a_2(t)\,\mathbf{j} + a_3(t)\,\mathbf{k},$$

then

$$\int_{t_0}^{T} \mathbf{a}(t)\,dt = \mathbf{i}\int_{t_0}^{T} a_1(t)\,dt + \mathbf{j}\int_{t_0}^{T} a_2(t)\,dt + \mathbf{k}\int_{t_0}^{T} a_3(t)\,dt. \tag{3}$$

Example 2. Compute $\int_0^{\pi/2} \mathbf{a}(t)\,dt$, where $\mathbf{a}(t) = \mathbf{i}\cos t - \mathbf{j}\sin^2 t$.

Solution. By virtue of formula (3),

$$\int_0^{\pi/2} \mathbf{a}(t)\,dt = \mathbf{i}\int_0^{\pi/2} \cos t\,dt - \mathbf{j}\int_0^{\pi/2} \sin^2 t\,dt$$

$$= \mathbf{i}\sin t\Big|_0^{\pi/2} - \mathbf{j}\left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\Big|_0^{\pi/2} = \mathbf{i} - \frac{\pi}{4}\mathbf{j}.$$

Compute the following integrals:

32. $\int_0^{\pi} \mathbf{a}(t)\,dt$, where $\mathbf{a} = \sin^2 t\cos t\cdot\mathbf{i} + \cos^2 t\sin t\cdot\mathbf{j} + \mathbf{k}$.

33. $\int\limits_0^1 \mathbf{a}(t)\,dt$, where $\mathbf{a}=\frac{\mathbf{i}e^{-t/2}}{2}+\frac{\mathbf{j}e^{t/2}}{2}+\mathbf{k}e^t$.

34. $\int\limits_0^1 \mathbf{a}(t)\,dt$, where $\mathbf{a}=3\pi\cos\pi t\cdot\mathbf{i}-\frac{\mathbf{j}}{1+t}+2t\mathbf{k}$.

35. $\int\limits_0^\pi \mathbf{a}(t)\,dt$, where $\mathbf{a}=(2t+\pi)\,\mathbf{i}+t\sin t\cdot\mathbf{j}+\pi\mathbf{k}$.

Example 3. An electric current I flows upwards along an infinite wire that coincides with the z-axis. Find the vector $\mathbf{H}$ of the magnetic field intensity set up by this current at an arbitrary point M (x, y, z) of space (Fig. 6).

Solution. We consider a sufficiently small element $PP_1 = d\zeta$ of the z-axis. By the Biot-Savart law, the intensity $d\mathbf{H}$ of the magnetic field set up at point M by the current flowing through element $d\zeta$ of the wire coincides in direction with the vector product $[d\boldsymbol{\zeta}, \mathbf{r}_1]$, where $d\boldsymbol{\zeta} = \mathbf{PP}_1$, $|d\boldsymbol{\zeta}| = d\zeta$, $\mathbf{r}_1 = \mathbf{PM}$ (see Fig. 6). By this same law, the modulus of the vector $d\mathbf{H}$ is

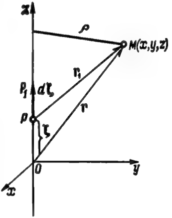

Fig. 6

$$|d\mathbf{H}|=\frac{I}{r_1^2}\sin\,(\widehat{d\boldsymbol{\zeta}, \mathbf{r}_1})\,d\zeta,$$

where $(\widehat{d\boldsymbol{\zeta}, \mathbf{r}_1})$ is the angle formed by the vectors $d\boldsymbol{\zeta}$ and $\mathbf{r}_1$.

Since

$$|[d\boldsymbol{\zeta}, \mathbf{r}_1]| = r_1 d\zeta \sin\,(\widehat{d\boldsymbol{\zeta}, \mathbf{r}_1}),$$

we can write

$$d\mathbf{H}=\frac{I}{r_1^3}[d\boldsymbol{\zeta}, \mathbf{r}_1]. \tag{4}$$

In order to obtain the desired vector **H** at the point M we have to sum all vectors $d\mathbf{H}$ pertaining to distinct elements PP_1 of the wire, that is, we have to integrate the expression (4) over the whole z-axis:

$$\mathbf{H}=\int\limits_{-\infty}^{+\infty}\frac{I}{r_1^3}[d\boldsymbol{\zeta},\mathbf{r}_1]. \tag{5}$$

We have

$$\mathbf{r}_1=\mathbf{OM}-\mathbf{OP}.$$

But

$$\mathbf{OM}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k},\qquad \mathbf{OP}=\zeta\mathbf{k},$$

and therefore

$$\mathbf{r}_1=x\mathbf{i}+y\mathbf{j}+(z-\zeta)\,\mathbf{k}$$

so that

$$r_1=|\mathbf{r}_1|=\sqrt{x^2+y^2+(z-\zeta)^2}=\sqrt{\rho^2+(z-\zeta)^2},$$

where $\rho=\sqrt{x^2+y^2}$ is the distance of point M from the axis of the wire.

For the vector product $[d\zeta,\mathbf{r}_1]$ we have

$$[d\boldsymbol{\zeta},\mathbf{r}_1]=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&d\zeta\\x&y&z-\zeta\end{vmatrix}=-\mathbf{i}y\,d\zeta+\mathbf{j}x\,d\zeta,$$

and formula (5) takes the form [the point $M(x, y, z)$ is fixed, $I=$ constant]

$$\mathbf{H}=I(-y\mathbf{i}+x\mathbf{j})\int\limits_{-\infty}^{+\infty}\frac{d\zeta}{[\rho^2+(z-\zeta)^2]^{3/2}}. \tag{6}$$

To compute the integral on the right-hand member of (6), make the substitution

$$\zeta-z=\rho\tan t,\quad d\zeta=\frac{\rho\,dt}{\cos^2 t}.$$

We then have

$$\int_{-\infty}^{+\infty} \frac{d\zeta}{[\rho^2+(z-\zeta)^2]^{3/2}} = \int_{-\pi/2}^{\pi/2} \frac{\rho\, dt}{\cos^2 t\,[\rho^2+\rho^2 \tan^2 t]^{3/2}}$$

$$= \frac{1}{\rho^2} \int_{-\pi/2}^{\pi/2} \cos t\, dt = \frac{2}{\rho^2}.$$

Thus, the intensity vector **H** of the magnetic field is in our case given by

$$\mathbf{H} = \frac{2I}{\rho^2}(-y\mathbf{i} + x\mathbf{j})$$

or

$$\mathbf{H} = \frac{2}{\rho^2}[\mathbf{I}, \mathbf{r}],$$

where $\mathbf{I} = I \cdot \mathbf{k}$ is the current vector, **r** is the radius vector of point $M(x, y, z)$ of the field, and ρ is the distance of M to the axis of the wire.

Example 4. *The motion of an electron in a homogeneous magnetic field.*

1°. Suppose a magnetic field **H** is set up in some region of space; let it be constant in magnitude and direction (a homogeneous field). Suppose at time $t = t_0$, an electron enters the field with an initial velocity $\mathbf{v}_0$. Determine the path the electron will take.

Solution. First suppose the vector $\mathbf{v}_0$ is perpendicular to **H** and that the initial position of the electron is at point M_0. Choose the origin O at an arbitrary point of the plane P passing through M_0 at right angles to the vector **H** (Fig. 7). Let the initial radius vector $\mathbf{OM}_0$ be $\mathbf{r}_0$, let **r** be the radius vector of the electron at the current instant of time t, and let **v** be the instantaneous velocity at that instant. The basic differential equation of motion is

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}.$$

It will be recalled that the force **F** acting at time t on the electron by the magnetic field is

$$\mathbf{F} = -e_0[\mathbf{H}, \mathbf{v}],$$

where e_0 is the absolute value of the electron charge. Thus,

$$m\frac{d^2\mathbf{r}}{dt^2} = e_0[\mathbf{v}, \mathbf{H}]. \tag{7}$$

At every instant t, the force $\mathbf{F}$ is perpendicular to the direction of velocity and to the direction of the field $\mathbf{H}$;

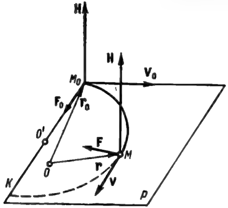

Fig. 7

at every instant, it will force the electron to deviate from a rectilinear path and to describe a certain curvilinear trajectory.

Let us rewrite (7) as

$$m\frac{d\mathbf{v}}{dt} = e_0\left[\frac{d\mathbf{r}}{dt}, \mathbf{H}\right]$$

and integrate from t_0 to t with respect to t. This yields

$$m\mathbf{v} - m\mathbf{v}_0 = e_0[\mathbf{r}, \mathbf{H}] - e_0[\mathbf{r}_0, \mathbf{H}]$$

or

$$m\mathbf{v} = e_0[\mathbf{r}, \mathbf{H}] + (m\mathbf{v}_0 - e_0[\mathbf{r}_0, \mathbf{H}]). \tag{8}$$

Now choose a coordinate origin O' such that the term in parentheses in the right-hand member of (8) vanishes, that is, so that

$$e_0[\mathbf{r}_0, \mathbf{H}] = m\mathbf{v}_0. \tag{9}$$

From (9) it follows that the initial vector $\mathbf{r}_0$ must be perpendicular to the vector $\mathbf{v}_0$ and must lie on the straight line M_0K, which is perpendicular to the plane of the vectors $\mathbf{v}_0$ and $\mathbf{H}$. By virtue of (9), the modulus of the vector $\mathbf{r}_0$ must satisfy the relation

$$e_0 \mid \mathbf{r}_0 \mid \cdot \mid \mathbf{H} \mid = m \mid \mathbf{v}_0 \mid,$$

whence

$$|\mathbf{r}_0| = \frac{m\,|\mathbf{v}_0|}{e_0\,|\mathbf{H}|}. \tag{10}$$

This determines the position of the new origin O'. Relative to this origin, equation (8) is rewritten thus:

$$m\mathbf{v} = e_0\,[\mathbf{r},\ \mathbf{H}] \tag{11}$$

or

$$m\,\frac{d\mathbf{r}}{dt} = e_0\,[\mathbf{r},\mathbf{H}]. \tag{12}$$

From equation (11) it follows that the trajectory of the electron is a plane curve lying in the plane P because at every instant the vector $\mathbf{v}$ is perpendicular to $\mathbf{H}$. Now take the scalar product of both sides of (12) by $\mathbf{r}$:

$$m\left(\mathbf{r},\ \frac{d\mathbf{r}}{dt}\right) = e_0\,(\mathbf{r},[\mathbf{r},\mathbf{H}]). \tag{13}$$

The mixed product in the right-hand member of (13) is zero, so that

$$\left(\mathbf{r},\ \frac{d\mathbf{r}}{dt}\right) = 0,$$

whence

$$\frac{d}{dt}(\mathbf{r}^2) = 0 \text{ or } \frac{d}{dt}(r^2) = 0, \text{ that is, } r^2 = \text{constant}.$$

This is the equation of a circle lying in the plane P with centre at the chosen point O'. The radius of the circle is found from formula (10) since the initial point M_0 must also lie on that circle. Thus, we finally have

$$r = r_0 = \frac{m\,|\mathbf{v}_0|}{e_0\,|\mathbf{H}|}. \tag{10}$$

Thus, if an electron enters a homogeneous magnetic field $\mathbf{H}$ with an initial velocity $\mathbf{v}_0$ at right angles to $\mathbf{H}$, then it will describe, in that field, a circular trajectory lying in the plane P perpendicular to $\mathbf{H}$ and passing through the initial point. The radius of the circle is

given by formula (10) and its centre O' lies on the straight line perpendicular to the plane of the vectors $\mathbf{v}_0$ and $\mathbf{H}$; note that a rotation from $\mathbf{v}_0$ to $\mathbf{H}$ must be seen from point O' as a counterclockwise rotation.

From (10) it is evident that the radius r_0 of the circle is inversely proportional to $|\mathbf{H}|$. Thus, the greater the intensity of the magnetic field, the greater the curvature of the trajectory.

From formula (11),

$$m\mathbf{v} = e_0 [\mathbf{r}, \mathbf{H}],$$

it is clear that if $\mathbf{r}$ is constant in modulus and is all the itme perpendicular to $\mathbf{H}$, then also the velocity $\mathbf{v}$ of the

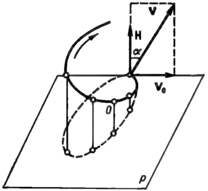

Fig. 8

point will be constant in magnitude,

$$|\mathbf{v}| = v_0 = \text{constant},$$

so that the electron is in uniform motion in the orbit. The period of revolution T is

$$T = \frac{2\pi r_0}{v_0} = 2\pi \frac{m}{e_0 |\mathbf{H}|}. \qquad (14)$$

This formula does not involve the initial velocity $\mathbf{v}_0$. Thus, irrespective of the initial velocity $\mathbf{v}_0$ which is perpendicular to $\mathbf{H}$ with which the electron enters the

homogeneous magnetic field **H**, it will perform a single orbital revolution and always in the same time T.

2°. Now suppose an electron enters a homogeneous magnetic field **H** with some initial velocity **V** that is not perpendicular to the vector **H**. This velocity may then be resolved into two components: the vector $\mathbf{v}_0$ at right angles to the field, and a vector $\mathbf{v}_1$ parallel to the magnetic field.

From the formula

$$\mathbf{F} = e\,[\mathbf{V},\ \mathbf{H}] = e_0\,[\mathbf{v}_0,\ \mathbf{H}]$$

it is evident that the "twisting" force **F** is given only by the perpendicular component $\mathbf{v}_0$ and that it imparts to the electron a rotational motion about the circle (centred at O') discussed above. As for the other component $\mathbf{v}_1$, the electron will retain it by inertia and, besides having a uniform circular motion, it will have a rectilinear and uniform motion in the direction of **H** with a velocity $v_1 = |\mathbf{V}| \cos\alpha$. The combination of these motions yields a helical curve with axis parallel to the vector **H** and passing through the point O' (Fig. 8).

Sec. 5. The first and second derivatives of a vector with respect to the arc length of a curve. The curvature of a curve. The principal normal

Consider a curve L in space. On it, choose a point M_0 as the origin and also choose a direction along L that will be regarded as positive. For a parameter, take the arc length s reckoned from M_0 of the curve (Fig. 9). Then the radius vector of a point M of the curve is

$$\mathbf{r} = \mathbf{r}\,(s).$$

With that choice of parameter,

$$\frac{d\mathbf{r}}{ds} = \boldsymbol{\tau}^0,$$

where $\boldsymbol{\tau}^0$ is a unit vector directed along the tangent to the curve L in the direction of increasing values of the parameter s.

If the vector $\mathbf{r}$ is given by the coordinates

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

then

$$\boldsymbol{\tau}^0 = \frac{dx}{ds}\,\mathbf{i} + \frac{dy}{ds}\,\mathbf{j} + \frac{dz}{ds}\,\mathbf{k}$$

and

$$\sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} = 1.$$

Since $|\boldsymbol{\tau}^0| = 1$, the vector $d\boldsymbol{\tau}^0/ds$ is orthogonal to the vector $\boldsymbol{\tau}^0$.

The modulus of the vector $d\boldsymbol{\tau}^0/ds$ is

$$\left|\frac{d\boldsymbol{\tau}^0}{ds}\right| = K.$$

Here, K is the curvature of the curve L at the point M.

The straight line having the direction of the vector $d\boldsymbol{\tau}^0/ds$ and passing through the point M of the curve is

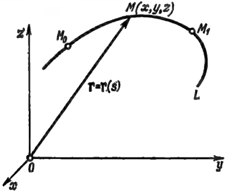

Fig. 9

termed the *principal normal* of the curve at the point M. Denoting the unit vector of that direction by $\mathbf{n}^0$, we have

$$\frac{d\boldsymbol{\tau}^0}{ds} = K\mathbf{n}^0. \tag{1}$$

The inverse of the curvature of a curve at a given point is called the *radius of curvature* of the curve at that point and is denoted by R:

$$R = \frac{1}{K}.$$

Thus, formula (1) may be rewritten as

$$\frac{d^2\mathbf{r}}{ds^2} = \frac{d\mathbf{r}^0}{ds} = \frac{\mathbf{n}^0}{R}.$$

From this,

$$K = \frac{1}{R} = \left| \frac{d^2\mathbf{r}}{ds^2} \right|$$

or

$$K = \frac{1}{R} = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}. \qquad (2)$$

Using (2), we can compute the curvature of a curve at any point if the curve is specified by parametric equations in which the parameter is the arc length s.

In the particular case of a plane curve, a circle of radius a,

$$\left.\begin{aligned} x &= a \cos \frac{s}{a}, \\ y &= a \sin \frac{s}{a}, \end{aligned}\right\}$$

we have

$$\frac{d^2x}{ds^2} = -\frac{1}{a} \cos \frac{s}{a}, \quad \frac{d^2y}{ds^2} = -\frac{1}{a} \sin \frac{s}{a}$$

and formula (2) yields

$$K = \frac{1}{R} = \sqrt{\frac{1}{a^2} \cos^2 \frac{s}{a} + \frac{1}{a^2} \sin^2 \frac{s}{a}} = \frac{1}{a}.$$

This means that the curvature of a circle of radius a is constant and is equal to the inverse of the radius of the circle.

If the curve L is given by the vector-parametric equation $\mathbf{r} = \mathbf{r}(t)$, where the parameter t is arbitrary, then

$$K = \frac{1}{R} = \frac{\left|\left[\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}\right]\right|}{\left|\frac{d\mathbf{r}}{dt}\right|^3}. \tag{3}$$

Formula (3) permits computing the curvature of the curve at any point provided we have an arbitrary parametric specification of that curve.

Example 1. Compute the curvature of the helical curve

$$\mathbf{r} = a\cos t\cdot\mathbf{i} + a\sin t\cdot\mathbf{j} + ht\mathbf{k}.$$

Solution. Since

$$\frac{d\mathbf{r}}{dt} = -a\sin t\cdot\mathbf{i} + a\cos t\cdot\mathbf{j} + h\mathbf{k},$$

$$\frac{d^2\mathbf{r}}{dt^2} = -a\cos t\cdot\mathbf{i} - a\sin t\cdot\mathbf{j},$$

the vector product

$$\left[\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}\right] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & h \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= ah\sin t\cdot\mathbf{i} - ah\cos t\cdot\mathbf{j} + a^2\mathbf{k}.$$

Consequently,

$$\left|\left[\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}\right]\right| = a\sqrt{a^2+h^2}, \quad \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{a^2+h^2}.$$

By virtue of (3),

$$K = \frac{1}{R} = \frac{a}{a^2+h^2}$$

or

$$R = \frac{a^2+h^2}{a} = \text{constant}.$$

Thus, a helical curve has a constant radius of curvature.

Find the radius of curvature of each of the given curves:

36. $\mathbf{r} = \ln\cos t\cdot\mathbf{i} + \ln\sin t\cdot\mathbf{j} + \sqrt{2}t\cdot\mathbf{k}$.

37. $\mathbf{r} = t^2\mathbf{i} + 2t^3\mathbf{j}$.

38. $\mathbf{r} = 3t^2\mathbf{i} + (3t - t^3)\,\mathbf{j} + 2\mathbf{k}$ for $t = 1$.
39. $\mathbf{r} = a\,(\cos t + t \sin t)\,\mathbf{i} + a\,(\sin t - t \cos t)\,\mathbf{j}$ for $t = \pi/2$.
40. $\mathbf{r} = a \cosh t \cdot \mathbf{i} + a \sinh t \cdot \mathbf{j} + at\mathbf{k}$ at any point t.

Sec. 6. Osculating plane. Binormal. Torsion. The Frenet formulas

The plane passing through the tangent line and principal normal to a given curve L at a point M is termed the *osculating plane* at the point M.

For a plane curve, the osculating plane coincides with the plane of the curve.

If the vector $\mathbf{r} = \mathbf{r}\,(t)$ has a continuous derivative $d\mathbf{r}/dt$ in the neighbourhood of a point t_0 and, besides, a second derivative $d^2\mathbf{r}\,(t_0)/dt^2$ such that

$$\left[\frac{d\mathbf{r}\,(t_0)}{dt},\ \frac{d^2\mathbf{r}\,(t_0)}{dt^2}\right] \neq 0,$$

then at the point $t = t_0$ there is an osculating plane to the curve $\mathbf{r} = \mathbf{r}\,(t)$ whose vector equation is

$$\left(\boldsymbol{\rho} - \mathbf{r}\,(t_0),\ \left[\frac{d\mathbf{r}\,(t_0)}{dt},\ \frac{d^2\mathbf{r}\,(t_0)}{dt^2}\right]\right) = 0,$$

where $\boldsymbol{\rho} = \boldsymbol{\rho}\,(t)$ is the radius vector of the current point of the plane.

The normal to the curve at the point M, which normal is perpendicular to the osculating plane of the curve at that point, is called the *binormal* of the curve at the given point M.

Denote by $\mathbf{b}^0$ the unit vector of the binormal oriented so that the vectors $\boldsymbol{\tau}^0$, $\mathbf{n}^0$, $\mathbf{b}^0$ form a right-handed trihedral (Fig. 10). Then

$$\mathbf{b}^{0^2} = 1, \quad \mathbf{b}^0 = [\boldsymbol{\tau}^0, \mathbf{n}^0].$$

For the derivative $d\mathbf{b}^0/ds$ we get

$$\frac{d\mathbf{b}^0}{ds} = \left[\boldsymbol{\tau}^0, \frac{d\mathbf{n}^0}{ds}\right].$$

The vector $d\mathbf{b}^0/ds$ is perpendicular both to the vector $\boldsymbol{\tau}^0$ and to the vector $\mathbf{b}^0$, that is, it is collinear with the vec-

tor $\mathbf{n}^0$. Set

$$\left|\frac{d\mathbf{b}^0}{ds}\right| = \frac{1}{T}.$$

We then have

$$\frac{d\mathbf{b}^0}{ds} = \frac{1}{T}\mathbf{n}^0.$$

The quantity $1/T$ is termed the *torsion* of the given curve, and T is called the *radius of torsion* of the curve.

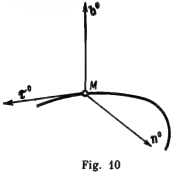

Fig. 10

The torsion of a curve is given by the formula

$$\frac{1}{T} = R^2\left(\frac{d\mathbf{r}}{ds}, \frac{d^2\mathbf{r}}{ds^2}, \frac{d^3\mathbf{r}}{ds^3}\right),$$

where the symbol $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ denotes a mixed product of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, that is, $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a}, [\mathbf{b}, \mathbf{c}])$.

For the case where the curve is given by the vector-parametric equation $\mathbf{r} = \mathbf{r}(t)$, we have

$$\frac{1}{T} = \frac{\left(\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \frac{d^3\mathbf{r}}{dt^3}\right)}{\left|\left[\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}\right]\right|^2}. \tag{1}$$

Example 1. Find the torsion of the helical curve

$$\mathbf{r} = a\cos t\cdot\mathbf{i} + a\sin t\cdot\mathbf{j} + ht\mathbf{k}.$$

Solution. We find the derivatives of the given vector:

$$\frac{d\mathbf{r}}{dt} = -a\sin t\cdot\mathbf{i} + a\cos t\cdot\mathbf{j} + h\mathbf{k},$$

$$\frac{d^2\mathbf{r}}{dt^2} = -a\cos t\cdot\mathbf{i} - a\sin t\cdot\mathbf{j},$$

$$\frac{d^3\mathbf{r}}{dt^3} = a\sin t\cdot\mathbf{i} - a\cos t\cdot\mathbf{j}.$$

The mixed product of these vectors is

$$\left(\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \frac{d^3\mathbf{r}}{dt^3}\right) = \begin{vmatrix} -a\sin t & a\cos t & h \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix} = a^2h.$$

In example 1, Sec. 5, we found that

$$\left|\left[\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}\right]\right|^2 = a^2(a^2+h^2).$$

Using (1), we obtain for the torsion

$$\frac{1}{T} = \frac{h}{a^2+h^2}.$$

Thus, the torsion of a helical curve is the same at all its points.

Example 2. Write the equation of the osculating plane at the point $t = 0$ of the helical curve

$$\mathbf{r} = a\cos t\cdot\mathbf{i} + a\sin t\cdot\mathbf{j} + ht\mathbf{k}.$$

Solution. We find the values of the derivatives of the given vector and its derivatives $d\mathbf{r}/dt$ and $d^2\mathbf{r}/dt^2$ at the point $t = 0$:

$$\mathbf{r}(0) = a\mathbf{i}, \quad \frac{d\mathbf{r}(0)}{dt} = a\mathbf{j} + h\mathbf{k}, \quad \frac{d^2\mathbf{r}(0)}{dt^2} = -a\mathbf{i}.$$

Consequently (see example 1, Sec. 5),

$$\left[\frac{d\mathbf{r}(0)}{dt}, \frac{d^2\mathbf{r}(0)}{dt^2}\right] = -ah\mathbf{j} + a^2\mathbf{k}.$$

The vector equation of the osculating plane is

$$\left(\boldsymbol{\rho} - \mathbf{r}(0), \frac{d\mathbf{r}(0)}{dt}, \frac{d^2\mathbf{r}(0)}{dt^2}\right) = 0$$

or

$$(\boldsymbol{\rho} - a\mathbf{i}, -ah\mathbf{j} + a^2\mathbf{k}) = 0.$$

Since the radius vector of the current point of the osculating plane $\rho = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, it follows that by passing to coordinate notation we obtain an equation of the desired plane in the form $hy - az = 0$.

Formulas expressing the derivatives of the vectors $\boldsymbol{\tau}^0$, $\mathbf{b}^0$, $\mathbf{n}^0$ are called *Frenet formulas*:

$$\frac{d\boldsymbol{\tau}^0}{ds} = \frac{1}{R}\mathbf{n}^0, \quad \frac{d\mathbf{b}^0}{ds} = \frac{1}{T}\mathbf{n}^0, \quad \frac{d\mathbf{n}^0}{ds} = -\frac{1}{R}\boldsymbol{\tau}^0 - \frac{1}{T}\mathbf{b}^0.$$

41. Write down the equation of the osculating plane at the point $t = 2$ of the curve

$$\mathbf{r} = t\mathbf{i} - t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}.$$

42. Write down the equation of the osculating plane at the point $t = 0$ of the curve

$$\mathbf{r} = e^t\mathbf{i} + e^{-t}\mathbf{j} + \sqrt{2}\,t\mathbf{k}.$$

43. Find the torsion at the point $t = 0$ of the curve

$$\mathbf{r} = e^t \cos t \cdot \mathbf{i} + e^t \sin t \cdot \mathbf{j} + e^t\mathbf{k}.$$

44. Find the torsion at any point t of the curve

$$r = a \cosh t \cdot \mathbf{i} + a \sinh t \cdot \mathbf{j} + at\mathbf{k}.$$

CHAPTER II

SCALAR FIELDS

Sec. 7. Examples of scalar fields. Level surfaces and level lines

Definition. If a value of a certain quantity is defined at every point of space or a portion of space, then we say that the *field* of the given quantity has been specified.

The field is termed a *scalar field* if the quantity in question is a scalar quantity, that is, if it is fully described by its numerical value.

Examples of scalar fields are: a temperature field, an electrostatic field.

Specifying a scalar field is accomplished by specifying the scalar function of a point M:

$$u = f(M).$$

If a Cartesian coordinate system xyz is introduced in space, we have

$$u = f(x, y, z).$$

Geometrically, a scalar field is characterized by a level surface; this is a locus of points at which the scalar function of the field assumes the same value. The level surface of a given field is defined by the equation

$$f(x, y, z) = C, \quad \text{where } C = \text{constant}.$$

In the case of a temperature field set up in a homogeneous and isotropic medium by a point source of heat, the level surfaces are spheres centred at the source (this is a central-symmetric field).

In the case of an infinite uniformly heated wire, the level surfaces (isothermic surfaces) are circular cylinders whose axes coincide with that of the wire.

Example 1. Construct the level surfaces of the scalar field

$$u = x + 2y + 3z.$$

Solution. The level surfaces are given by the equation

$$x + 2y + 3z = C, \quad \text{where } C = \text{constant}.$$

This is a one-parameter family of parallel planes.

Example 2. Find the level surfaces of the scalar field

$$u = x^2 + y^2 - z^2.$$

Solution. The level surfaces are given by the equation

$$x^2 + y^2 - z^2 = C, \quad \text{where } C = \text{constant}.$$

For $C = 0$, we obtain a circular cone. For any $C > 0$, we obtain a hyperboloid of revolution of one sheet with the axis coincident with the z-axis. For $C < 0$, we obtain a hyperboloid of revolution of two sheets.

Example 3. Find the level surfaces of the scalar field

$$u = \arcsin \frac{z}{\sqrt{x^2+y^2}}.$$

Solution. The domain of definition of the given scalar field is found from the inequality

$$\left|\frac{z}{\sqrt{x^2+y^2}}\right| \leqslant 1, \text{ that is, } 0 \leqslant \frac{z^2}{x^2+y^2} \leqslant 1,$$

whence $0 \leqslant z^2 \leqslant x^2 + y^2$. This double inequality shows that the field is defined outside a circular cone $z^2 = x^2 + y^2$ and on it, with the exception of its vertex $O\ (0, 0, 0)$.

The level surfaces are found from the equation

$$\arcsin \frac{z}{\sqrt{x^2+y^2}} = C, \text{ where } -\frac{\pi}{2} \leqslant C \leqslant \frac{\pi}{2}.$$

That is, $z/\sqrt{x^2+y^2} = \sin C$ or $z^2 = (x^2 + y^2)\sin^2 C$. This is a family of circular cones located outside the cone $z^2 = x^2 + y^2$ with a common axis of symmetry, Oz, and a common vertex, $O\ (0, 0, 0)$, at which the given field

is not defined; note that the cone itself, $z^2 = x^2 + y^2$, is also included in the family.

Example 4. Find the level surfaces of the scalar field

$$u = e^{(\mathbf{a},\, \mathbf{r})},$$

where **a** is a constant vector and **r** is the radius vector of a point.

Solution. Here,

$$\mathbf{r} = \{x,\ y,\ z\} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and let

$$\mathbf{a} = \{a_1,\ a_2,\ a_3\} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

Then the scalar product

$$(\mathbf{a},\ \mathbf{r}) = a_1x + a_2y + a_3z.$$

The equation of the level surfaces is

$$e^{(\mathbf{a},\, \mathbf{r})} = C, \quad C > 0,$$

whence

$$(\mathbf{a},\ \mathbf{r}) = \ln C$$

or

$$a_1x + a_2y + a_3z = \ln C.$$

This is a family of parallel planes.

Find the level surfaces of the following scalar fields:

45. $u = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16}$.

46. $u = x^2 + y^2 - z$.

47. $u = \frac{x^2 + y^2}{z}$.

48. $u = 2y^2 + 9z^2$.

49. $u = 3^{x+2y-z}$.

50. $u = \frac{(\mathbf{a},\ \mathbf{r})}{(\mathbf{b},\ \mathbf{r})}$ (**a**, **b** are constant vectors).

51. $u = \ln |\mathbf{r}|$.

52. $u = e^{(\mathbf{a},\, \mathbf{b},\, \mathbf{r})}$ (**a**, **b** are constant vectors).

A scalar field is said to be *plane* if there is a plane such that in all planes parallel to the given plane the scalar field is the same.

If we take this plane as the xy-plane, then the scalar field is given by the scalar function

$$u = f(x, y),$$

which is to say it is not dependent on z.

An example of a plane scalar field is the temperature field of an infinite, uniformly heated wire.

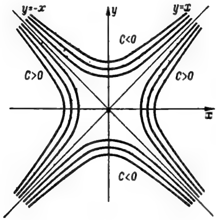

Fig. 11

Geometrically, plane scalar fields are characterized by level lines; these are loci in which the scalar function has one and the same value.

Example 5. Find the level lines of the scalar field

$$u = x^2 - y^2.$$

Solution. The level lines are given by the equations

$$x^2 - y^2 = C, \quad \text{where } C = \text{constant}.$$

When $C = 0$, we obtain a pair of straight lines:

$$y = x, \quad y = -x.$$

For $C \neq 0$, we obtain a family of hyperbolas (Fig. 11).

Find the level lines of the following plane fields:

53. $u=2x-y$.

54. $u=\ln\sqrt{\frac{y}{2x}}$.

55. $u=\frac{y^2}{x}$.

56. $u=e^{x^2-y^2}$.

57. Find the level lines of the scalar field u given implicitly by the equation

$$u+x\ln u+y=0.$$

Sec. 8. Directional derivative

Suppose we have a scalar field defined by a scalar function

$$u=f(M).$$

In the field, take a point M_0 and choose a direction indicated by the vector $\mathbf{l}$. Then in the field take another point M so that the vector $\mathbf{M_0M}$ is parallel to $\mathbf{l}$. Denote by Δu the difference

$$\Delta u=f(M)-f(M_0)$$

and by Δl the length of the vector $\mathbf{M_0M}$. The ratio $\Delta u/\Delta l$ defines the average rate of change of the scalar field per unit of length in the given direction. Allow the point M to move towards the point M_0 so that the vector $\mathbf{M_0M}$ is always collinear with the vector $\mathbf{l}$. Then $\Delta l\to 0$.

Definition. If, as $\Delta l\to 0$, there is a limit to the ratio $\Delta u/\Delta l$, then it is called the *derivative* of the function $u=f(M)$ at the given point M_0 in the direction of $\mathbf{l}$ and is denoted by the symbol $\partial u/\partial l$ so that, by definition, we have

$$\frac{\partial u}{\partial l}=\lim_{\Delta l\to 0}\frac{\Delta u}{\Delta l}=\lim_{\Delta l\to 0}\frac{f(M)-f(M_0)}{\Delta l},\quad \mathbf{M_0M}\parallel\mathbf{l}.$$

This definition of a directional derivative is invariant, that is, it is not connected with any choice of coordinate system.

Suppose a Cartesian coordinate system has been introduced in space and suppose the function $f(M) = f(x, y, z)$ is differentiable at the point $M_0(x_0, y_0, z_0)$. Then

$$\left.\frac{\partial u}{\partial l}\right|_{M_0} = \left.\frac{\partial u}{\partial x}\right|_{M_0} \cos\alpha + \left.\frac{\partial u}{\partial y}\right|_{M_0} \cos\beta + \left.\frac{\partial u}{\partial z}\right|_{M_0} \cos\gamma, \qquad (1)$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$, the direction cosines of the vector

$$\mathbf{l} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$

are found from the formulas

$$\cos\alpha = \frac{a_1}{|\mathbf{l}|}\ \cos\beta = \frac{a_2}{|\mathbf{l}|}, \quad \cos\gamma = \frac{a_3}{|\mathbf{l}|},$$

$$|\mathbf{l}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

The symbols $\left.\frac{\partial u}{\partial x}\right|_{M_0}, \left.\frac{\partial u}{\partial y}\right|_{M_0}, \left.\frac{\partial u}{\partial z}\right|_{M_0}$ signify that the partial derivatives are taken at the point M_0.

For a plane field $u = f(x, y)$, the directional derivative $\mathbf{l}$ at the point $M_0(x_0, y_0)$ is

$$\left.\frac{\partial u}{\partial l}\right|_{M_0} = \left.\frac{\partial u}{\partial x}\right|_{M_0} \cos\alpha + \left.\frac{\partial u}{\partial y}\right|_{M_0} \sin\alpha, \qquad (2)$$

where α is the angle formed by the vector $\mathbf{l}$ and the x-axis.

Remark. The partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial u/\partial z$ themselves are derivatives of the function u in the direction of the coordinate axes Ox, Oy, Oz respectively.

Formula (1)—used to compute the directional derivative at a given point—holds true even when the point M tends to M_0 along a curve for which the vector $\mathbf{l}$ is the tangent line at the point M_0.

Example 1. Find the derivative of the scalar field

$$u = xyz$$

at the point $M_0(1, -1, 1)$ in the direction from M_0 to $M_1(2, 3, 1)$.

Solution. We find the direction cosines of the vector $\mathbf{M_0M_1} = \{1, 4, 0\}$, the length of which is $|\mathbf{M_0M_1}| = \sqrt{17}$,

and we have

$$\cos\alpha = \frac{1}{\sqrt{17}}, \quad \cos\beta = \frac{4}{\sqrt{17}}, \quad \cos\gamma = 0.$$

The values of the partial derivatives of the function $u = xyz$ at the point $M_0\,(1, -1, 1)$ are

$$\left.\frac{\partial u}{\partial x}\right|_{M_0} = -1, \quad \left.\frac{\partial u}{\partial y}\right|_{M_0} = 1, \quad \left.\frac{\partial u}{\partial z}\right|_{M_0} = -1.$$

Using formula (1), we get

$$\left.\frac{\partial u}{\partial l}\right|_{M_0} = -\frac{1}{\sqrt{17}} + \frac{4}{\sqrt{17}} - 1\cdot 0 = \frac{3}{\sqrt{17}}.$$

The fact that $\left.\frac{\partial u}{\partial l}\right|_{M_0} > 0$ means that the scalar field at M_0 increases in the given direction.

Example 2. Compute the derivative of the scalar field

$$u = \arctan xy$$

at the point $M_0\,(1, 1)$, which belongs to the parabola $y = x^2$, in the direction of the curve (in the direction of increasing abscissas).

Solution. The direction of **l** of the parabola $y = x^2$ at the point $M_0\,(1, 1)$ is the direction of the tangent to the parabola at that point (Fig. 12).

Suppose the tangent **l** to the curve at M_0 forms with the x-axis an angle α. We then have

$$y' = 2x, \quad \tan\alpha = y'\,|_{x=1} = 2,$$

whence the direction cosines of the tangent line are

$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{5}}, \quad \cos\beta = \sin\alpha$$
$$= \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = \frac{2}{\sqrt{5}}.$$

The values of the partial derivatives of the given function $u\,(x, y)$ at the point $M_0\,(1, 1)$ are

$$\left.\frac{\partial u}{\partial x}\right|_{M_0} = \left.\frac{1}{1+x^2y^2}\right|_{M_0} = \frac{1}{2}, \quad \left.\frac{\partial u}{\partial y}\right|_{M_0} = \left.\frac{1}{1+x^2y^2}\right|_{M_0} = \frac{1}{2}.$$

Substituting them into (2), we obtain

$$\frac{\partial u}{\partial l}=\frac{1}{2}\cdot\frac{1}{\sqrt{5}}+\frac{1}{2}\cdot\frac{2}{\sqrt{5}}=\frac{3}{2\sqrt{5}}.$$

Example 3. Find the derivative of the scalar field

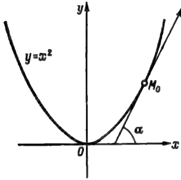

Fig. 12

$u = xz^2 + 2yz$ at the point M_0 (1, 0, 2) along the circle

$$\left.\begin{aligned} x&=1+\cos t,\\ y&=\sin t-1,\\ z&=2.\end{aligned}\right\}$$

Solution. The vector equation of the circle is of the form

$$\mathbf{r}(t) = (1+\cos t)\,\mathbf{i} + (\sin t - 1)\,\mathbf{j} + 2\mathbf{k}.$$

We find the vector $\boldsymbol{\tau}$ tangent to it at any point M to be

$$\boldsymbol{\tau}=\frac{d\mathbf{r}}{dt}=-\sin t\cdot\mathbf{i}+\cos t\cdot\mathbf{j}.$$

The given point M_0 (1, 0, 2) is found in the xz-plane in the first octant and is associated with the value of the parameter $t=\pi/2$. At this point we have

$$\boldsymbol{\tau}|_{M_0}=-\sin\frac{\pi}{2}\cdot\mathbf{i}+\cos\frac{\pi}{2}\cdot\mathbf{j}=-\mathbf{i}.$$

From this we obtain that the direction cosines of the tangent to the circle are equal to $\cos\alpha = -1$, $\cos\beta = 0$, $\cos\gamma = 0$. The values of the partial derivatives of the given scalar field at the point M_0 (1, 0, 2) are:

$$\left.\frac{\partial u}{\partial x}\right|_{M_0} = z^2\Big|_{M_0} = 4, \quad \left.\frac{\partial u}{\partial y}\right|_{M_0} = 2z\Big|_{M_0} = 4,$$

$$\left.\frac{\partial u}{\partial z}\right|_{M_0} = (2xz + 2y)\Big|_{M_0} = 4.$$

Hence the desired derivative is

$$\left.\frac{\partial u}{\partial l}\right|_{M_0} = \left.\frac{\partial u}{\partial \tau}\right|_{M_0} = 4\cdot(-1) + 4\cdot 0 + 4\cdot 0 = -4.$$

In the following problems, it is required to find, for the given functions, the derivative at the point M_0 (x_0, y_0, z_0) in the direction of the point M_1 (x_1, y_1, z_1).

58. $u = \sqrt{x^2+y^2+z^2}$, $M_0(1, 1, 1)$, $M_1(3, 2, 1)$.

59. $u = x^2y + xz^2 - 2$, $M_0(1, 1, -1)$, $M_1(2, -1, 3)$.

60. $u = xe^y + ye^x - z^2$, $M_0(3, 0, 2)$, $M_1(4, 1, 3)$.

61. $u = \frac{x}{y} - \frac{y}{x}$, $M_0(1, 1)$, $M_1(4, 5)$.

62. Find the derivative of the scalar field

$$u = \ln(x^2 + y^2)$$

at the point M_0 (1, 2) of the parabola $y^2 = 4x$ in the direction of the curve.

63. Find the derivative of the scalar field $u = \arctan y/x$ at the point M_0 (2, −2) of the circle $x^2 + y^2 - 4x = 0$ along an arc of the circle.

64. Find the derivative of the scalar field $u = x^2 + y^2$ at the point M_0 (x_0, y_0) of the circle $x^2 + y^2 = R^2$ in the direction of the circle.

65. Find the derivative of the scalar field $u = 2xy + y^2$ at the point $(\sqrt{2}, 1)$ of the ellipse $x^2/4 + y^2/2 = 1$ in the direction of the outer normal to the ellipse at that point.

66. Find the derivative of the scalar field $u = x^2 - y^2$ at the point (5, 4) of the hyperbola $x^2 - y^2 = 9$ in the direction of the curve.

67. Find the derivative of the scalar field $u = \ln(xy + yz + xz)$ at the point $M_0(0, 1, 1)$ in the direction of the circle $x = \cos t$, $y = \sin t$, $z = 1$.

68. Find the derivative of the scalar field $u = x^2 + y^2 + z^2$ at the point M_0 that corresponds to the value of the parameter $t = \pi/2$ in the direction of the helical curve $x = R\cos t$, $y = R\sin t$, $z = at$.

Sec. 9. The gradient of a scalar field

Suppose we have a scalar field defined by a scalar function

$$u = f(x, y, z)$$

where the function f is assumed to be differentiable.

Definition. The *gradient* of a scalar field u at a given point M is a vector denoted by the symbol grad u and defined by the equation

$$\operatorname{grad} u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}. \tag{1}$$

Using formula (1) of Sec. 8 for the directional derivative, we have

$$\frac{\partial u}{\partial l} = (\operatorname{grad} u, \mathbf{l}^0), \tag{2}$$

where $\mathbf{l}^0$ is a unit vector in the direction of $\mathbf{l}$, that is,

$$\mathbf{l}^0 = \frac{\mathbf{l}}{|\mathbf{l}|} = \mathbf{i}\cos\alpha + \mathbf{j}\cos\beta + \mathbf{k}\cos\gamma.$$

Properties of a gradient

1. The gradient is in the direction of the normal to the level surface (or to the level line if the field is a plane field).

2. The gradient is in the direction of increasing values of the function of the field.

3. The modulus of the gradient is equal to the largest directional derivative at a given point of the field:

$$\max\frac{\partial u}{\partial l} = |\operatorname{grad} u| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2}.\,^*$$

* The maximum is taken over all directions of $\mathbf{l}$ at the given point of the field.

These properties yield an invariant characteristic of the gradient. They state that the vector grad u indicates the direction and magnitude of maximum change of a scalar field at a given point.

Example 1. Find the gradient of the scalar field

$$u = x - 2y + 3z.$$

Solution. By (1) we have

$$\operatorname{grad} u = 1\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}.$$

The level surfaces of the given scalar field are the planes $x - 2y + 3z = C$; the vector grad $u = \{1, -2, 3\}$ is the normal vector of the planes of this family.

Example 2. Find the greatest steepness (rate) of rise of the surface $u = x^y$ at the point $M(2, 2, 4)$.

Solution. We have

$$\operatorname{grad} u = yx^{y-1}\mathbf{i} + x^y \ln x\mathbf{j}, \qquad \operatorname{grad} u\,|_M = 4\mathbf{i} + 4\ln 2\mathbf{j},$$

$$\left(\frac{\partial u}{\partial l}\right)_{\max} = |\operatorname{grad} u| = 4\sqrt{1 + (\ln 2)^2}.$$

Example 3. Find the unit vector of the normal to the level surface of the scalar field

$$u = x^2 + y^2 + z^2.$$

Solution. The level surfaces of the given scalar field are the spheres

$$x^2 + y^2 + z^2 = C \quad (C > 0).$$

The gradient is directed along the normal to the level surface so that grad $u = 2x\cdot\mathbf{i} + 2y\cdot\mathbf{j} + 2z\cdot\mathbf{k}$ defines the vector of the normal to the level surface at the point $M(x, y, z)$. For the unit vector of the normal, we obtain the expression

$$\mathbf{n}^0 = \frac{\operatorname{grad} u}{|\operatorname{grad} u|} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{\mathbf{r}}{|\mathbf{r}|}.$$

Example 4. Find the gradient of the field $u = (\mathbf{a}, \mathbf{b}, \mathbf{r})$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and $\mathbf{r}$ is the radius vector of the point.

Solution. Let

$$\mathbf{a} = \{a_1, a_2, a_3\}, \quad \mathbf{b} = \{b_1, b_2, b_3\}, \quad \mathbf{r} = \{x, y, z\}.$$

Then

$$u = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ x & y & z \end{vmatrix}.$$

By the rule for differentiating a determinant* we have

$$\frac{\partial u}{\partial x} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 1 & 0 & 0 \end{vmatrix}, \quad \frac{\partial u}{\partial y} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 1 & 0 \end{vmatrix}, \quad \frac{\partial x}{\partial z} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 1 \end{vmatrix}.$$

Hence

$$\operatorname{grad} u = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = [\mathbf{a}, \mathbf{b}].$$

---

* Given a determinant $D(t)$ whose elements a_{ij} are differentiable functions of t:

$$D(t) = \begin{vmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \dots & \dots & \dots & \dots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{vmatrix}.$$

Then the derivative of the determinant, $D'(t)$, is found from the formula

$$D'(t) = \begin{vmatrix} a'_{11}(t) & a'_{12}(t) & \dots & a'_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \dots & \dots & \dots & \dots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{vmatrix} + \begin{vmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a'_{21}(t) & a'_{22}(t) & \dots & a'_{2n}(t) \\ \dots & \dots & \dots & \dots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{vmatrix} + \dots + \begin{vmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \dots & \dots & \dots & \dots \\ a'_{n1}(t) & a'_{n2}(t) & \dots & a'_{nn}(t) \end{vmatrix}.$$

Example 5. Find the gradient of the distance

$$r=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2},$$

where $P(x, y, z)$ is the point of the field being studied and $P_0(x_0, y_0, z_0)$ is some fixed point.

Solution. We have

$$\operatorname{grad} r=\frac{\partial r}{\partial x}\mathbf{i}+\frac{\partial r}{\partial y}\mathbf{j}+\frac{\partial r}{\partial z}\mathbf{k}$$

$$=\frac{(x-x_0)\,\mathbf{i}+(y-y_0)\,\mathbf{j}+(z-z_0)\,\mathbf{k}}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}=\mathbf{r}^0,$$

which is the unit vector of the direction $\mathbf{P}_0\mathbf{P}$.

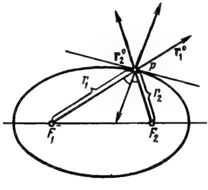

Fig. 13

Example 6. Let us consider the scalar function

$$u=r_1+r_2,$$

where r_1, r_2 are the distances of some point $P(x, y)$ of the plane from two fixed points, F_1 and F_2, of the plane.

Solution. The level lines of this function are ellipses. We have (see example 5)

$$\operatorname{grad}(r_1+r_2)=\mathbf{r}_1^0+\mathbf{r}_2^0.$$

This shows that the gradient is equal to the diagonal of a rhombus constructed on the unit vectors of the radius vectors drawn to point P from the foci F_1 and F_2 (Fig. 13). Consequently, the normal to the ellipse at some point

bisects the angle between the radius vectors drawn to that point.

Physical interpretation: a light ray coming from one focus enters the other focus.

Example 7. Find the angle θ between the gradients of the functions

$$u=\sqrt{x^2+y^2} \text{ and } v=x+y+2\sqrt{xy}$$

at the point $M_0\,(1, 1)$.

Solution. We find the gradients of the given functions at the point $M_0\,(1, 1)$:

$$\operatorname{grad} u\,|_{M_0}=\left.\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}\right|_{M_0}=\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}},$$

$$\operatorname{grad} v\,|_{M_0}=\left.\left[\left(1+\sqrt{\frac{y}{x}}\right)\mathbf{i}+\left(1+\sqrt{\frac{x}{y}}\right)\mathbf{j}\right]\right|_{M_0}=2\mathbf{i}+2\mathbf{j}.$$

The angle θ between grad u and grad v at the point M_0 is found from

$$\cos\theta=\frac{(\operatorname{grad} u,\ \operatorname{grad} v)\,|_{M_0}}{|\operatorname{grad} u\,|_{M_0}\cdot|\operatorname{grad} v\,|_{M_0}}=\frac{\frac{2}{\sqrt{2}}+\frac{2}{\sqrt{2}}}{1\cdot 2\sqrt{2}}=1.$$

From this we have

$$\theta=0.$$

Example 8. Find the directional derivative of the radius vector $\mathbf{r}$ for the function $u=\sin r$, where $r=|\mathbf{r}|$.

Solution. By (2), the directional derivative of the given function of the radius vector $\mathbf{r}$ is

$$\frac{\partial u}{\partial r}=(\operatorname{grad}\sin r,\ \mathbf{r}^0). \tag{3}$$

We find the gradient of the function:

$$\begin{aligned}\operatorname{grad}\sin r&=\frac{\partial(\sin r)}{\partial x}\mathbf{i}+\frac{\partial(\sin r)}{\partial y}\mathbf{j}+\frac{\partial(\sin r)}{\partial z}\mathbf{k}\\&=\frac{d(\sin r)}{dr}\frac{\partial r}{\partial x}\mathbf{i}+\frac{d(\sin r)}{dr}\frac{\partial r}{\partial y}\mathbf{j}+\frac{d(\sin r)}{dr}\frac{\partial r}{\partial z}\mathbf{k}\\&=\left(\frac{\partial r}{\partial x}\mathbf{i}+\frac{\partial r}{\partial y}\mathbf{j}+\frac{\partial r}{\partial z}\mathbf{k}\right)\cos r=\mathbf{r}^0\cos r.\end{aligned} \tag{4}$$

Substituting (4) into (3), we get

$$\frac{\partial u}{\partial r} = (\mathbf{r}^0 \cos r, \mathbf{r}^0) = (\mathbf{r}^0, \mathbf{r}^0) \cos r = \cos r.$$

Example 9. Find the derivative of the scalar field $u = f(x, y, z)$ at the point $M_0(x_0, y_0, z_0)$ of the curve l specified by the system of equations

$$\left.\begin{array}{l} f(x, y, z) = a, \\ \varphi(x, y, z) = 0 \end{array}\right\}, \quad a = \text{constant},$$

in the direction of the curve.

Solution. The direction of the curve l is given by the direction of its tangent vector $\boldsymbol{\tau}$, which, by definition, is a vector that is tangent to the surface $f(x, y, z) = a$. The surface $f(x, y, z) = a$ is a level surface of the given scalar field $u = f(x, y, z)$. Since

$$\frac{\partial u}{\partial l} = (\operatorname{grad} u, \mathbf{l}^0) = (\operatorname{grad} u, \boldsymbol{\tau}^0)$$

and the vector grad u is perpendicular to the level surface $f(x, y, z) = a$, it follows that grad u is perpendicular to the unit vector $\boldsymbol{\tau}^0$, and therefore

$$\left.\frac{\partial u}{\partial l}\right|_{M_0} = (\operatorname{grad} u, \boldsymbol{\tau}^0)\,|_{M_0} = 0.$$

Example 10. At the point $M_0(1, 1, 1)$, find the direction of the greatest change in the scalar field $u = xy + yz + xz$ and the magnitude of that change.

Solution. The direction of the greatest change of the field is indicated by the vector grad $u(M)$. We find it thus:

$$\operatorname{grad} u(M) = (y + z)\,\mathbf{i} + (x + z)\,\mathbf{j} + (y + x)\,\mathbf{k}$$

and hence grad $u(M_0) = 2(\mathbf{i} + \mathbf{j} + \mathbf{k})$. This vector determines the direction of the greatest increase in the given field at the point $M_0(1, 1, 1)$. The magnitude of maximum change of the field at this point is

$$\max \frac{\partial u}{\partial l} = |\operatorname{grad} u(M_0)| = 2\sqrt{3}.$$

69. Find the gradient of the scalar field $u = \ln(x^2 + y^2 + z^2)$ at the point $M_0(1, 1, -1)$.
70. Find the gradient of the scalar field $u = ze^{x^2+y^2+z^2}$ at the point $O(0, 0, 0)$.

71. Find the angle φ between the gradients of the function $u = \arctan x/y$ at the points M_1 (1, 1) and M_2 (−1, −1).
72. Find the angle φ between the gradients of the function $u = (x + y)\, e^{x+y}$ at the points M_1 (0, 0) and M_2 (1, 1).
73. Find the angle φ between the gradients of the functions $u = \sqrt{x^2 + y^2 + z^2}$ and $v = \ln (x^2 + y^2 + z^2)$ at the point M_0 (0, 0, 1).
74. Find the points at which the gradient of the scalar field $u = \sin (x + y)$ is equal to $\mathbf{i} + \mathbf{j}$.
75. Find the points at which the modulus of the gradient of the scalar field $u = \ln \sqrt{x^2 + y^2 + z^2}$ is equal to unity.
76. Let $u = u (x, y, z)$ and $v = v (x, y, z)$ be functions differentiable at the point $M (x, y, z)$. Show that

(a) $\operatorname{grad} (\lambda u) = \lambda \operatorname{grad} u$, $\lambda =$ constant;
(b) $\operatorname{grad} (u \pm v) = \operatorname{grad} u \pm \operatorname{grad} v$;
(c) $\operatorname{grad} (uv) = v \operatorname{grad} u + u \operatorname{grad} v$;
(d) $\operatorname{grad}\left(\frac{u}{v}\right) = \frac{v \operatorname{grad} u - u \operatorname{grad} v}{v^2}$, $v \neq 0$.

77. Show that

$$\operatorname{grad} u(\varphi) = \frac{du}{d\varphi} \operatorname{grad} \varphi,$$

where $\varphi = \varphi (x, y, z)$ is a differentiable function and $u = u (\varphi)$ has a derivative with respect to φ.

Find the gradients of the following scalar fields if

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \quad r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2},$$

and $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.
78. $u = \ln r$.
79. $u = (\mathbf{a}, \mathbf{r})$.
80. $u = (\mathbf{a}, \mathbf{r}) \cdot (\mathbf{b}, \mathbf{r})$.
81. $u = |[\mathbf{a}, \mathbf{r}]|^2$.
82. Show that

$$(\operatorname{grad} u (r), \mathbf{r}) = u' (r) \cdot r.$$

83. Show that

$$[\operatorname{grad} u (r), \mathbf{r}] = 0.$$

84. Let $w = f (u, v)$, where $u = u (x, y, z)$, $v = v (x, y, z)$.

Prove that

$$\operatorname{grad} w = \frac{\partial f}{\partial u} \operatorname{grad} u + \frac{\partial f}{\partial v} \operatorname{grad} v$$

if f, u, v are differentiable functions.

85. Suppose G is a convex region in space (that is, a region such that if two points M and N belong to G, then the whole line segment MN lies in G). Let there be given in G a scalar field $u(M)$ which at all points has a gradient that is continuous and bounded in G:

$$|\operatorname{grad} u(M)| \leqslant A, \quad M \in G, \quad A = \text{constant}.$$

Prove that for any points M and N of G we have the inequality

$$|u(N) - u(M)| \leqslant A\,|\mathbf{MN}|.$$

86. Find the derivative of the function $u = x^2/a^2 + y^2/b^2 + z^2/c^2$ at an arbitrary point $M(x, y, z)$ in the direction of the radius vector $\mathbf{r}$ of that point.

87. Find the derivative of the function $u = 1/r$, where $r = |\mathbf{r}|$ in the direction of the vector $\mathbf{l} = \cos\alpha\cdot\mathbf{i} + \cos\beta\cdot\mathbf{j} + \cos\gamma\cdot\mathbf{k}$. Under what condition is the derivative equal to zero?

88. Find the derivative of the function $u = 1/r$, where $r = |\mathbf{r}|$, in the direction of its gradient.

89. Find the derivative of the function $u = yze^x$ at the point $M_0(0, 0, 1)$ in the direction of its gradient.

90. Find the derivative of the scalar field

$$u = u(x, y, z)$$

in the direction of the gradient of the scalar field

$$v = v(x, y, z).$$

Under what condition is it equal to zero?

91. For the following scalar fields, find the direction and magnitude of greatest change at the given points M_0:

(a) $u(M) = x^2y + y^2z + z^2x$; $M_0(1, 0, 0)$.

(b) $u(M) = xyz$; $M_0(2, 1, -1)$.

CHAPTER III

VECTOR FIELDS

Sec. 10. Vector lines. Differential equations of vector lines

Definition 1. We say that a *vector field* is given if a vector quantity $\mathbf{a} = \mathbf{a}(M)$ is specified at each point M of space or of a portion of space.

If a Cartesian coordinate system is introduced in the space, then specifying the vector field $\mathbf{a} = \mathbf{a}(M)$ is equivalent to specifying three scalar functions of the point $P(M)$, $Q(M)$, $R(M)$ so that

$$\mathbf{a}(M) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}.$$

Definition 2. A *vector line* of a vector field $\mathbf{a}$ is a curve at each point M of which the vector $\mathbf{a}$ is directed along the tangent to the curve.

Let a vector field be defined by the vector

$$\mathbf{a} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k},$$

where

$$P = P(x, y, z), \quad Q = Q(x, y, z), \quad R = R(x, y, z)$$

are continuous functions of x, y, z that have bounded partial derivatives of the first order.

Then the differential equations of the vector lines are of the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{1}$$

Integrating this system of two differential equations (1) yields a system of two finite equations:

$$\varphi_1(x, y, z) = C_1, \quad \varphi_2(x, y, z) = C_2$$

which, taken together, define a two-parameter family of vector lines:

$$\left.\begin{aligned}\varphi_1(x, y, z) = C_1,\\ \varphi_2(x, y, z) = C_2.\end{aligned}\right\} \tag{2}$$

If the conditions of the theorem of existence and uniqueness of a solution are fulfilled in a certain region G for system (1), then a unique vector line

$$\left.\begin{aligned}\varphi_1(x, y, z) = \varphi_1(x_0, y_0, z_0),\\ \varphi_2(x, y, z) = \varphi_2(x_0, y_0, z_0)\end{aligned}\right\}$$

passes through every point $M_0(x_0, y_0, z_0) \in G$.

Example 1. Find the vector lines of the vector field

$$\mathbf{a} = [\mathbf{c}, \mathbf{r}],$$

where $\mathbf{c}$ is a constant vector.

Solution. We have

$$\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}, \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

so that

$$\mathbf{a} = [\mathbf{c}, \mathbf{r}] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_1 & c_2 & c_3 \\ x & y & z \end{vmatrix}$$

$$= (c_2z - c_3y)\,\mathbf{i} + (c_3x - c_1z)\,\mathbf{j} + (c_1y - c_2x)\,\mathbf{k}.$$

The differential equations of the vector lines are

$$\frac{dx}{c_2z - c_3y} = \frac{dy}{c_3x - c_1z} = \frac{dz}{c_1y - c_2x}. \tag{3}$$

Multiply the numerator and denominator of the first fraction by x, the second by y, the third by z and add termwise. Using a property of proportions, we have

$$\frac{dx}{c_2z - c_3y} = \frac{dy}{c_3x - c_1z} = \frac{dz}{c_1y - c_2x} = \frac{x\,dx + y\,dy + z\,dz}{0},$$

whence

$$x\,dx + y\,dy + z\,dz = 0$$

and this means that

$$x^2 + y^2 + z^2 = A_1, \quad A_1 = \text{constant} > 0.$$

Now, multiplying the numerator and denominator of the first fraction of (3) by c_1, the second by c_2, the third by c_3

and adding termwise, we obtain

$$\frac{dx}{c_2z-c_3y}=\frac{dy}{c_3x-c_1z}=\frac{dz}{c_1y-c_2x}=\frac{c_1\,dx+c_2\,dy+c_3\,dz}{0},$$

whence

$$c_1\,dx+c_2\,dy+c_3\,dz=0$$

and, consequently,

$$c_1x+c_2y+c_3z=A_2,\quad A_2=\text{constant}.$$

The desired equations of the vector lines are:

$$\left.\begin{aligned}x^2+y^2+z^2&=A_1,\\ c_1x+c_2y+c_3z&=A_2.\end{aligned}\right\}$$

These equations show that the vector lines are obtained via the intersection of spheres (having a common centre at the origin of coordinates) with planes perpendicular to the vector $\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$. From this it follows that the vector lines are circles whose centres lie on a straight line passing through the coordinate origin in the direction of the vector $\mathbf{c}$. The planes of the circles are perpendicular to the indicated straight line (Fig. 14).

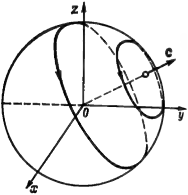

Fig. 14

Example 2. Find the vector line of the field

$$\mathbf{a}=-y\mathbf{i}+x\mathbf{j}+b\mathbf{k},$$

which line passes through the point (1, 0, 0).

Solution. The differential equations of the vector lines are

$$\frac{dx}{-y}=\frac{dy}{x}=\frac{dz}{b},$$

whence we find

$$x^2+y^2=C_1,\quad C_1>0$$

or, introducing the parameter t,

$$x=\sqrt{C_1}\cos t,\quad y=\sqrt{C_1}\sin t.$$

In this case, the equation

$$\frac{dy}{x}=\frac{dz}{b}$$

takes the form

$$\frac{\sqrt{C_1}\cos t\,dt}{\sqrt{C_1}\cos t}=\frac{dz}{b}\quad\text{or}\quad dz=b\,dt,$$

whence we obtain

$$z=bt+C_2.$$

Thus, the parametric equations of the vector lines are

$$\left.\begin{aligned} x&=\sqrt{C_1}\cos t,\\ y&=\sqrt{C_1}\sin t,\\ z&=bt+C_2.\end{aligned}\right\}\qquad(4)$$

If we require that the vector line pass through the point (1, 0, 0), we will have

$$1=\sqrt{C_1}\cos t,\quad 0=\sqrt{C_1}\sin t,\quad 0=bt+C_2.$$

The first two equations of this system are satisfied for $t=2k\pi$, $k=0, \pm1, \ldots$ and for $C_1=1$. Taking $k=0$, we get $t=0$ and the last equation of the system yields $C_2=0$. The desired vector line passing through the point (1, 0, 0) is

$$\left.\begin{aligned} x&=\cos t,\\ y&=\sin t,\\ z&=bt.\end{aligned}\right\}$$

This is a helical curve.

Find the vector lines of the following vector fields:

92. $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

93. $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, where a_1, a_2, a_3 are constants.

94. $\mathbf{a}=(z-y)\,\mathbf{i}+(x-z)\,\mathbf{j}+(y-x)\,\mathbf{k}$.

95. Find the vector line of the field

$$\mathbf{a}=x^2\mathbf{i}-y^3\mathbf{j}+z^2\mathbf{k},$$

which line passes through the point $(^1/_2,\ -^1/_2,\ 1)$.

A vector field is said to be *plane* if all the vectors of **a** are located in parallel planes and the field is the same in each of the planes.

If a Cartesian coordinate system xOy is introduced in one of the planes, the vectors of the field will not contain any components along the z-axis and the coordinates of a vector will be independent of z, that is,

$$\mathbf{a} = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}.$$

The differential equations of the vector lines of a plane field are of the form

$$\frac{dx}{P(x, y)} = \frac{dy}{Q(x, y)} = \frac{dz}{0}$$

or

$$\left.\begin{aligned} \frac{dy}{dx} &= \frac{Q(x, y)}{P(x, y)}, \\ z &= \text{constant}. \end{aligned}\right\}$$

From this it is evident that the vector lines of a plane field are plane curves lying in planes parallel to the xy-plane.

Example 3. Find the vector lines of a magnetic field of an infinite current conductor.

Solution. We will assume the conductor is in the direction of the z-axis and that the current I flows in that direction. The intensity vector $\mathbf{H}$ of the magnetic field set up by the current is

$$\mathbf{H} = \frac{2}{\rho^2}[\mathbf{I}, \mathbf{r}], \tag{5}$$

where $\mathbf{I} = I \cdot \mathbf{k}$ is the current vector, $\mathbf{r}$ is the radius vector of the point $M(x, y, z)$, and ρ is the distance from the axis of the wire to the point M. Expanding the vector product (5), we obtain

$$\mathbf{H} = -\frac{2Iy}{\rho^2}\,\mathbf{i} + \frac{2Ix}{\rho^2}\,\mathbf{j}.$$

The differential equations of the vector lines are

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dz}{0},$$

whence

$$\left.\begin{aligned} x^2+y^2&=R^2,\\ z&=C.\end{aligned}\right\}$$

That is, the vector lines are circles with centres on the z-axis (Fig. 15).

Find the vector lines of the following plane vector fields:

96. $\mathbf{a} = x\mathbf{i} + 2y\mathbf{j}$.

97. $\mathbf{a} = x\mathbf{i} + z\mathbf{k}$.

98. $\mathbf{a} = x\mathbf{i} - y\mathbf{j}$.

99. $\mathbf{a} = 2z\mathbf{j} + 4y\mathbf{k}$.

100. $\mathbf{a} = x^2\mathbf{i} + y^2\mathbf{j}$.

101. $\mathbf{a} = z\mathbf{j} - y\mathbf{k}$.

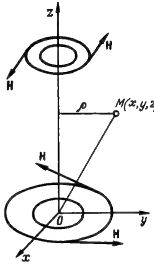

Fig. 15

The differential equations of the vector lines

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

may be written as

$$\frac{dx}{dt}=P,\quad \frac{dy}{dt}=Q,\quad \frac{dz}{dt}=R$$

or, in vector form, as

$$\frac{d\mathbf{r}}{dt}=\mathbf{a}(M). \qquad (6)$$

This form of the equations of vector lines turns out to be convenient in the solution of a number of problems.

Example 4. Find the vector lines of the field $\mathbf{a} = [\mathbf{c}, \mathbf{r}]$, where $\mathbf{c}$ is a constant vector.

Solution. Applying (6), we get

$$\frac{d\mathbf{r}}{dt}=[\mathbf{c}, \mathbf{r}]. \qquad (7)$$

Forming the scalar product of both sides of (7) by $\mathbf{c}$ and using the properties of a mixed product, we find

$$\left(\mathbf{c}, \frac{d\mathbf{r}}{dt}\right)=\frac{d}{dt}(\mathbf{c}, \mathbf{r})=0. \qquad (8)$$

Similarly, forming the scalar product of both sides of (7) by $\mathbf{r}$, we obtain

$$\left(\mathbf{r}, \frac{d\mathbf{r}}{dt}\right) = \frac{d}{dt}(\mathbf{r}, \mathbf{r}) = 0. \tag{9}$$

From equation (8) it follows that

$$(\mathbf{c}, \mathbf{r}) = \text{constant}$$

and from equation (9) it follows that

$$(\mathbf{r}, \mathbf{r}) = \text{constant}.$$

The vector lines are lines of intersection of the planes $(\mathbf{c}, \mathbf{r}) = \text{constant}$ with the spheres $\mathbf{r}^2 = \text{constant}$.

Find the vector lines of the following vector fields:

102. $\mathbf{a} = f(r) \cdot \mathbf{r}$.

103. $\mathbf{a} = (\mathbf{a}_0, \mathbf{r})\, \mathbf{b}_0$, where $\mathbf{a}_0$, $\mathbf{b}_0$ are constant vectors.

Sec. 11. The flux of a vector field. Methods of calculating flux

1. The flux of a vector field. Suppose we have a vector field

$\mathbf{a}(M) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$,
where the coordinates $P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$ of the vector $\mathbf{a}(M)$ are continuous [the field $\mathbf{a}(M)$ is continuous] in some region G. Let S be a smooth or piecewise smooth two-sided surface in which a definite side has been chosen (an oriented surface).

Definition. The flux Π of a vector field $\mathbf{a}(M)$ through an oriented surface S is defined as the surface integral of the first kind, over the surface S, of the projection of the vector $\mathbf{a}(M)$ by the normal $\mathbf{n}(M)$ to that surface:

$$\Pi = \iint_S \mathrm{pr}_{\mathbf{n}}\mathbf{a}\, dS = \iint_S (\mathbf{a}, \mathbf{n}^0)\, dS,$$

where $\mathbf{n}^0$ is the unit vector of the normal $\mathbf{n}$ to the chosen side of the surface S; dS is the area element of the surface $\cdot\, S$.

In the case of a closed surface, we will always choose the outer normal $\mathbf{n}$ that is directed outwards from the region bounded by the surface S.

If α, β, γ are the angles that the normal $\mathbf{n}$ forms with the coordinate axes Ox, Oy, Oz to the surface S, then the flux may be expressed in terms of a surface integral of the second kind:

$$\Pi = \iint_S (\mathbf{a}, \mathbf{n}^0)\, dS = \iint_S [P(x, y, z)\cos\alpha + Q(x, y, z)\cos\beta + R(x, y, z)\cos\gamma]\, dS$$

or

$$\Pi = \iint_S (\mathbf{a}, \mathbf{n}^0)\, dS = \iint_S P(x, y, z)\, dy\, dz + Q(x, y, z)\, dx\, dz + R(x, y, z)\, dx\, dy,$$

where

$$\cos\alpha\, dS = dy\, dz, \quad \cos\beta\, dS = dx\, dz, \quad \cos\gamma\, dS = dx\, dy.$$

Basic properties of the flux of a vector field

(a) The flux reverses sign when the orientation of the surface is changed (that is, when the orientation of the normal $\mathbf{n}$ to the surface S is changed):

$$\iint_{S^+} (\mathbf{a}, \mathbf{n}^0)\, dS = -\iint_{S^-} (\mathbf{a}, \mathbf{n}^0)\, dS,$$

where S^+ is the side of the surface S on which the normal $\mathbf{n}$ is chosen, and S^- is the side of S on which the normal $-\mathbf{n}$ is taken (see [7]).

(b) Linearity:

$$\iint_S (\lambda\mathbf{a} + \mu\mathbf{b}, \mathbf{n}^0)\, dS = \lambda \iint_S (\mathbf{a}, \mathbf{n}^0)\, dS + \mu \iint_S (\mathbf{b}, \mathbf{n}^0)\, dS,$$

where λ and μ are constant numbers.

(c) Additivity: if the surface S consists of several smooth parts $S_1, S_2, \ldots, S_m$, then the flux of the vector field $\mathbf{a}(M)$ through S is equal to the sum of the fluxes

of the vector **a** (M) through the surfaces $S_1, S_2, \ldots, S_m$:

$$\Pi = \sum_{k=1}^{m} \iint_{S_k} (\mathbf{a}, \mathbf{n}^0)\, dS.$$

This property permits extending the notion of flux to piecewise smooth surfaces.

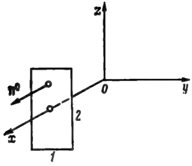

Fig. 16

Example 1. Find the flux of the vector $\mathbf{a} = \mathbf{i}$ through an area perpendicular to the x-axis and having the shape of a rectangle with sides *1* and *2* (Fig. 16) in the positive direction of the x-axis.

Solution. According to the definition of the flux of a vector through a surface S, we have

$$\Pi = \iint_S (\mathbf{a}, \mathbf{n}^0)\, dS.$$

In our case, $\mathbf{a} = \mathbf{i}$, $\mathbf{n}^0 = \mathbf{i}$ so that $(\mathbf{a}, \mathbf{n}^0) = (\mathbf{i}, \mathbf{i}) = 1$. Taking into account that the area of the rectangle is equal to 2, we obtain

$$\Pi = \iint_S 1\, dS = 2.$$

Remark. If we had chosen the unit vector of the normal to the area S so that $\mathbf{n}^0 = -\mathbf{i}$, we would have got $\Pi = -2$.

Example 2. Compute the flux of the vector field $\mathbf{a} = \mathbf{r}$, where $\mathbf{r}$ is the radius vector, through a right circular cylinder of altitude h, base radius R and the z-axis.

Solution. The surface S consists of a lateral surface σ_1, an upper base σ_2 and a lower base σ_3 of the cylinder. By the additivity property, the desired flux Π is equal to $\Pi = \Pi_1 + \Pi_2 + \Pi_3$, where Π_1, Π_2, Π_3 are the fluxes of the given field through σ_1, σ_2, σ_3 respectively.

On the lateral surface σ_1 of the cylinder the outer normal $\mathbf{n}^0$ is parallel to the xy-plane and therefore

$$(\mathbf{a}, \mathbf{n}^0) = (\mathbf{r}, \mathbf{n}^0) = \mathrm{pr}_{\mathbf{n}^0}\mathbf{r} = R$$

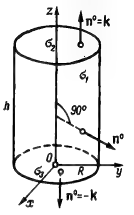

Fig. 17

Fig. 17). Hence

$$\Pi_1 = \iint_{\sigma_1} (\mathbf{a}_1, \mathbf{n}^0)\, dS = R \iint_{\sigma_1} dS = R \cdot 2\pi R h = 2\pi R^2 h.$$

On the upper base σ_2 the normal $\mathbf{n}^0$ is parallel to the z-axis and therefore we can put $\mathbf{n}^0 = \mathbf{k}$ (see Fig. 17). Then

$$(\mathbf{a}, \mathbf{n}^0) = (\mathbf{r}, \mathbf{k}) = \mathrm{pr}_{Oz}\,\mathbf{r} = h$$

and so

$$\Pi_2 = \iint_{\sigma_2} (\mathbf{a}, \mathbf{n}^0)\, dS = h \iint_{\sigma_2} dS = h \cdot \pi R^2 = \pi R^2 h.$$

On the lower base σ_3 the vector $\mathbf{a} = \mathbf{r}$ is perpendicular to the normal $\mathbf{n}^0 = -\mathbf{k}$. Therefore $(\mathbf{a}, \mathbf{n}^0) = (\mathbf{r}, -\mathbf{k}) = 0$ and

$$\Pi_3 = \iint_{\sigma_3} (\mathbf{a}, \mathbf{n}^0)\, dS = 0.$$

The desired flux is then

$$\Pi = \oint\!\!\oint_S (\mathbf{a}, \mathbf{n}^0)\, dS = 3\pi R^2 h.$$

Example 3. Find the flux of the vector field

$$\mathbf{a} = \frac{\mathbf{r}}{|\mathbf{r}|^3}$$

through a sphere of radius R with centre at the origin of coordinates.

Solution. Since the normal $\mathbf{n}$ to the sphere is collinear with the radius vector $\mathbf{r}$, we can take $\mathbf{n}^0 = \mathbf{r}^0 = \mathbf{r}/|\mathbf{r}|$. Therefore

$$(\mathbf{a}, \mathbf{n}^0) = \left(\frac{\mathbf{r}}{|\mathbf{r}|^3}, \frac{\mathbf{r}}{|\mathbf{r}|}\right) = \frac{1}{|\mathbf{r}|^4}(\mathbf{r}, \mathbf{r}) = \frac{|\mathbf{r}|^2}{|\mathbf{r}|^4} = \frac{1}{|\mathbf{r}|^2}.$$

But on the sphere S we have $|\mathbf{r}| = R$, and so $(\mathbf{a}, \mathbf{n}^0) = 1/R^2$.

The desired flux Π is

$$\Pi = \oiint_S (\mathbf{a}, \mathbf{n}^0)\, dS = \frac{1}{R^2} \oiint_S dS = 4\pi$$

since the area of the whole sphere S is equal to $\oiint_S dS = 4\pi R^2$.

104. Compute the flux of the vector $\mathbf{a} = 3\mathbf{j}$ through an area having the shape of a triangle with vertices at the points M_1 (1, 2, 0), M_2 (0, 2, 0), M_3 (0, 2, 2) in the direction of the coordinate origin.

105. Find the flux of the vector

$$\mathbf{a} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k},$$

where α, β, γ are constants, through an area perpendicular to the z-axis and having the shape of a circle of radius R, in the positive direction of the z-axis.

106. Find the flux of the vector $\mathbf{a} = \mathbf{r}$ through the outer side of a circular cone whose vertex lies at the origin of coordinates; the base radius is equal to R and the altitude is h (the axis of the cone is along the z-axis).

107. Find the flux of the vector $\mathbf{a} = f(|\mathbf{r}|)\,\mathbf{r}$ through a sphere of radius R with centre at the coordinate origin.

II. Methods of computing the flux of a vector.

1°. *The method of projection onto one of the coordinate planes.* Let an open surface S be projected one-to-one onto the xy-plane into a region D_{xy}. In this case, the

surface S may be given by the equation $z = f(x, y)$ and since the area element dS of the surface is

$$dS = \frac{dx\,dy}{|\cos\gamma|},$$

it follows that computing the flux Π through the chosen side of the surface S reduces to computing a double integral via the formula

$$\Pi = \iint_S (\mathbf{a}, \mathbf{n}^0)\,dS = \iint_{D_{xy}} \frac{(\mathbf{a}, \mathbf{n}^0)}{|\cos\gamma|}\bigg|_{z=f(x,\,y)} dx\,dy. \quad (1)$$

Here the unit vector $\mathbf{n}^0$ of the normal to the chosen side of the surface S is found from the formula

$$\mathbf{n}^0 = \pm\frac{\operatorname{grad}[z-f(x, y)]}{|\operatorname{grad}[z-f(x, y)]|} = \pm\frac{-\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}; \quad (2)$$

and $\cos\gamma$ is equal to the coefficient of the unit vector $\mathbf{k}$ in formula (2):

$$\cos\gamma = \pm\frac{1}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}. \quad (3)$$

If the angle γ between the z-axis and the normal $\mathbf{n}^0$ is acute, then in (2) and (3) the plus sign is taken, and if the angle γ is obtuse, the minus sign is taken. The symbol

$$\frac{(\mathbf{a}, \mathbf{n}^0)}{|\cos\gamma|}\bigg|_{z=f(x,\,y)}$$

signifies that $f(x, y)$ must be substituted for z in the integrand.

If it appears to be convenient to project the surface S onto the yz- and xz-planes, use is made of the following formulas to compute the flux Π:

$$\Pi = \iint_{D_{yz}} \frac{(\mathbf{a}, \mathbf{n}^0)}{|\cos\alpha|}\bigg|_{x=\varphi(y,\,z)} dy\,dz \quad (4)$$

or

$$\Pi = \iint_{D_{xz}} \frac{(\mathbf{a}, \mathbf{n}^0)}{|\cos\beta|}\bigg|_{y=\psi(x,\,z)} dx\,dz. \tag{5}$$

Formula (4) is used when the surface S is projected one-to-one into the region D_{yz} of the yz-plane, which means that it may be given by the equation $x = \varphi(y, z)$; $\cos\alpha$ is found as the coefficient of the unit vector $\mathbf{i}$ in the formula

$$\mathbf{n}^0 = \pm\frac{\operatorname{grad}[x-\varphi(y,\,z)]}{|\operatorname{grad}[x-\varphi(y,\,z)]|} = \pm\frac{\mathbf{i}-\frac{\partial\varphi}{\partial y}\mathbf{j}-\frac{\partial\varphi}{\partial z}\mathbf{k}}{\sqrt{1+\left(\frac{\partial\varphi}{\partial y}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2}}.$$

That is,

$$\cos\alpha = \pm\frac{1}{\sqrt{1+\left(\frac{\partial\varphi}{\partial y}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2}}.$$

The plus sign is taken if the angle α between the x-axis and the normal $\mathbf{n}^0$ is acute, and the minus sign if the angle α is obtuse.

Formula (5) is used in the case of a one-to-one projection of the surface S onto the xz-plane; in this case, S may be specified by the equation $y = \psi(x, z)$ and then

$$\mathbf{n}^0 = \pm\frac{\operatorname{grad}[y-\psi(x,\,z)]}{|\operatorname{grad}[y-\psi(x,\,z)]|} = \pm\frac{-\frac{\partial\psi}{\partial x}\mathbf{i}+\mathbf{j}-\frac{\partial\psi}{\partial z}\mathbf{k}}{\sqrt{1+\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial z}\right)^2}};$$

$\cos\beta$ is equal to the coefficient of the unit vector $\mathbf{j}$ in this formula, that is,

$$\cos\beta = \pm\frac{1}{\sqrt{1+\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2}}.$$

If the angle β between the y-axis and the normal $\mathbf{n}^0$ is acute, we take the plus sign, and if the angle β is obtuse, we take the minus sign.

Remark. When the surface S is specified implicitly by the equation $\Phi(x, y, z) = 0$, the unit vector of the

normal

$$\mathbf{n}^0 = \mathbf{i}\cos\alpha + \mathbf{j}\cos\beta + \mathbf{k}\cos\gamma$$

is found from the formula

$$\mathbf{n}^0 = \pm\frac{\operatorname{grad}\Phi(x, y, z)}{|\operatorname{grad}\Phi(x, y, z)|}$$

$$= \pm\frac{\frac{\partial\Phi}{\partial x}\mathbf{i}+\frac{\partial\Phi}{\partial y}\mathbf{j}+\frac{\partial\Phi}{\partial z}\mathbf{k}}{\sqrt{\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2}},$$

where the sign on the right is determined by the choice of the normal to the surface S.

To compute the flux Π of a vector field $\mathbf{a}$ through a surface S, it is necessary to project the surface one-to-one onto one of the xy-, xz-, yz-planes; this is possible if the equation $\Phi(x, y, z) = 0$ is uniquely solvable with respect to z ($z = f(x, y)$), y ($y = \psi(x, z)$) or x ($x = \varphi(y, z)$) respectively. Then take advantage of one of the formulas (1), (4), (5).

Example 4. Find the flux of the vector field

$$\mathbf{a} = (x - 2z)\,\mathbf{i} + (x + 3y + z)\,\mathbf{j} + (5x + y)\,\mathbf{k}$$

through the upper side of the triangle ABC with vertices at the points A (1, 0, 0), B (0, 1, 0), C (0, 0, 1).

Solution. The equation of the plane in which the triangle ABC lies is of the form $x + y + z = 1$, whence $z = 1 - x - y$. The triangle ABC is projected one-to-one onto the xy-plane into the region D_{xy}, which is the triangle OAB (Fig. 18).

It is given that the normal $\mathbf{n}^0$ to the plane in which the triangle ABC lies forms an acute angle γ with the z-axis and so we take the plus sign in (2) and obtain

$$\mathbf{n}^0 = \frac{\operatorname{grad}(x+y+z-1)}{|\operatorname{grad}(x+y+z-1)|} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}. \quad (6)$$

We find the scalar product

$$(\mathbf{a}, \mathbf{n}^0) = (x-2z)\frac{1}{\sqrt{3}} + (x+3y+z)\frac{1}{\sqrt{3}} + (5x+y)\frac{1}{\sqrt{3}} = \frac{7x+4y-z}{\sqrt{3}}.$$

From formula (6) we find that $\cos\gamma = 1/\sqrt{3} > 0$ and, hence,

$$dS = \frac{dx\,dy}{\cos\gamma} = \sqrt{3}\,dx\,dy.$$

Using formula (1), we compute the desired flux:

$$\Pi = \iint_S (\mathbf{a}, \mathbf{n}^0)\,dS = \iint_{D_{xy}} (7x + 4y - z)\,|_{z=1-x-y}\,dx\,dy$$

$$= \iint_{D_{xy}} (8x + 5y - 1)\,dx\,dy = \int_0^1 dx \int_0^{1-x} (8x + 5y - 1)\,dy = \frac{5}{3}.$$

Example 5. Find the flux of the vector $\mathbf{a} = y^2\mathbf{j} + z\mathbf{k}$ through the portion of the surface $z = x^2 + y^2$ cut off by

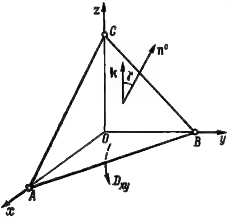

Fig. 18

the plane $z = 2$. The outer normal is taken with respect to the region bounded by the paraboloid.

Solution. The given surface (a paraboloid of revolution) is projected one-to-one onto the xy-plane into the circle D_{xy} (Fig. 19). We find the unit vector of the normal $\mathbf{n}^0$ to the surface S:

$$\mathbf{n}^0 = \pm\frac{\operatorname{grad}(z - x^2 - y^2)}{|\operatorname{grad}(z - x^2 - y^2)|} = \pm\frac{-2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}}.$$

It is given that the normal $\mathbf{n}^0$ forms an obtuse angle γ with the z-axis and therefore the minus sign is taken in front of the fraction. Thus,

$$\mathbf{n}^0 = \frac{2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}},$$

whence

$$\cos\gamma = -\frac{1}{\sqrt{4x^2 + 4y^2 + 1}} < 0$$

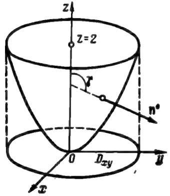

Fig. 19

and so

$$dS = \frac{dx\,dy}{|\cos\gamma|} = \sqrt{4x^2 + 4y^2 + 1}\,dx\,dy.$$

We find the scalar product

$$(\mathbf{a}, \mathbf{n}^0) = \frac{2y^3 - z}{\sqrt{4x^2 + 4y^2 + 1}}.$$

The desired flux is, by (1), equal to

$$\Pi = \iint_S (\mathbf{a}, \mathbf{n}^0)\,dS = \iint_{D_{xy}} (2y^3 - z)\,|_{z=x^2+y^2}\,dx\,dy$$

$$= \iint_{D_{xy}} (2y^3 - y^2 - x^2)\,dx\,dy.$$

The domain of integration D_{xy} is a circle of radius $R = \sqrt{2}$ with centre at the coordinate origin. Introducing polar coordinates $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, we have

$$\Pi = \iint\limits_{D_{xy}} (2\rho^3\sin^3\varphi - \rho^2)\,\rho\,d\rho\,d\varphi$$

$$= \int\limits_0^{2\pi} d\varphi \int\limits_0^{\sqrt{2}} (2\rho^4\sin^3\varphi - \rho^3)\,d\rho = -2\pi\frac{\rho^4}{4}\Big|_0^{\sqrt{2}} = -2\pi.$$

Example 6. Find the flux of the vector field $\mathbf{a} = \mathbf{i} - \mathbf{j} + xyz\mathbf{k}$ through the circle S obtained by cutting

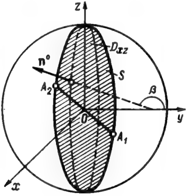

Fig. 20

the sphere $x^2 + y^2 + z^2 \leqslant R^2$ with a plane $y = x$. Take the side of the circle facing the positive portion of the x-axis.

Solution. Since the plane $y = x$ is perpendicular to the xy-plane, the circle S lying in that plane is projected onto the xy-plane into the line segment A_1A_2 and so the one-to-oneness of the projection is disrupted. The circle S is projected one-to-one onto the other coordinate planes. Projecting the circle onto the xz-plane, we obtain a region D_{xz} bounded by an ellipse (Fig. 20). The equation of the ellipse can be found by eliminating y from the system

of equations

$$\left.\begin{aligned} x^2+y^2+z^2=R^2, \\ y=x, \end{aligned}\right\}$$

whence

$$2x^2+z^2=R^2 \text{ or } \frac{x^2}{\frac{R^2}{2}}+\frac{z^2}{R^2}=1.$$

It is given that the normal to the circle S forms an obtuse angle β with the y-axis (see Fig. 20) and so we take

$$\mathbf{n}=-\operatorname{grad}(y-x)=\mathbf{i}-\mathbf{j},$$

$$\mathbf{n}^0=\frac{\mathbf{n}}{|\mathbf{n}|}=\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{j}.$$

From the latter equation we have $\cos\beta=-1/\sqrt{2}<0$. The area element dS of the circle is equal to

$$dS=\frac{dx\,dz}{|\cos\beta|}=\sqrt{2}\,dx\,dz.$$

We find the scalar product: $(\mathbf{a}, \mathbf{n}^0)=\sqrt{2}$.

The desired flux is, using formula (5),

$$\Pi=\iint\limits_{D_{xz}} 2\,dx\,dz=2\iint\limits_{D_{xz}} dx\,dz=2\cdot\frac{\pi R^2}{\sqrt{2}}=\sqrt{2}\,R^2\pi$$

since the area Q of the region D_{xz} bounded by an ellipse with semi-axes $a=R/\sqrt{2}$ and $b=R$ is equal to

$$Q=\iint\limits_{D_{xz}} dx\,dz=\pi ab=\frac{\pi R^2}{\sqrt{2}}.$$

Example 7. Compute the flux of the vector $\mathbf{a}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ through the outer side of the lateral surface of the circular cylinder $x^2+y^2=R^2$ bounded by the planes $z=0$ and $z=H$ $(H>0)$.

Solution. The given cylinder is projected onto the xy-plane into a line, namely, into the circle (Fig. 21)

$$\left.\begin{aligned} x^2+y^2=R^2, \\ z=0. \end{aligned}\right\}$$

We will therefore project the cylinder onto the other coordinate planes, for instance, the yz-plane. Since the cylinder does not project one-to-one onto the yz-plane, we take advantage of the additivity property of the flux of the vector and represent the desired flux Π as a sum of fluxes: $\Pi = \Pi_1 + \Pi_2$, where Π_1 is the flux of the field through the portion S_1 of the cylinder located in the region where $y \geqslant 0$, and Π_2 is the flux of that field through the portion S_2 of the cylinder located in the region where $y < 0$. On S_1 we have

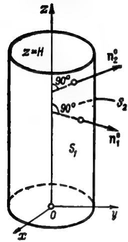

Fig. 21

$$\mathbf{n}^0 = \frac{x\mathbf{i} + y\mathbf{j}}{R}, \quad (\mathbf{a}, \mathbf{n}^0) = \frac{x^2 + y^2}{R} = R$$

and so

$$\Pi_1 = \iint_{S_1} R\, dS = R \iint_{S_1} dS = RS,$$

where S is the area of the portion S_1 of the cylinder. Since $S = \pi RH$, it follows that $\Pi_1 = \pi R^2 H$.

On S_2 we again have

$$\mathbf{n}^0 = \frac{x\mathbf{i} + y\mathbf{j}}{R}, \quad (\mathbf{a}, \mathbf{n}^0) = \frac{x^2 + y^2}{R} = R$$

and so

$$\Pi_2 = \iint_{S_2} R\, dS = RS = \pi R^2 H.$$

The desired flux is $\Pi = 2\pi R^2 H$.

Remark. The solution is made simpler if we introduce curvilinear coordinates $x = R\cos\varphi$, $y = R\sin\varphi$, $z = z$ on the cylinder (see item 3° below).

To find the flux of the vector field $\mathbf{a} = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$ through the surface S specified by the equation $z = f(x, y)$ by the method of projection onto a coordinate plane, it is not necessary to

find the unit vector of the normal $\mathbf{n}^0$, and we can take the vector

$$\mathbf{n} = \pm \operatorname{grad}[z - f(x, y)] = \pm \left(-\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}\right).$$

Formula (1) for finding the flux Π becomes

$$\Pi = \iint\limits_S (\mathbf{a}, \mathbf{n}^0)\, dS = \iint\limits_{D_{xy}} (\mathbf{a}, \mathbf{n})\Big|_{z=f(x,\, y)} dx\, dy. \tag{7}$$

In similar fashion we obtain formulas for computing the fluxes through surfaces given by the equation $x = \varphi(y, z)$ or $y = \psi(x, z)$.

Formula (7) is written thus in coordinate form:

$$\Pi = \pm \iint\limits_{D_{xy}} \left\{-P[x, y, f(x, y)]\frac{\partial f}{\partial x} - Q[x, y, f(x, y)]\frac{\partial f}{\partial y} + R[x, y, f(x, y)]\right\} dx\, dy.$$

Example 8. Compute the flux of the vector field

$$\mathbf{a} = x\mathbf{i} + y\mathbf{j} + \sqrt{x^2 + y^2 - 1}\,\mathbf{k}$$

through the outer side of the hyperboloid of one sheet $z = \sqrt{x^2 + y^2 - 1}$ bounded by the planes $z = 0$, $z = \sqrt{3}$.

Solution. The given surface is projected one-to-one onto the xy-plane into the region D_{xy} bounded by the circles

$$\left.\begin{aligned} x^2 + y^2 &= 1, \\ z &= 0, \end{aligned}\right\} \text{ and } \left.\begin{aligned} x^2 + y^2 &= 4, \\ z &= 0. \end{aligned}\right\}$$

We find the outer normal $\mathbf{n}$:

$$\mathbf{n} = \pm \operatorname{grad}\left(z - \sqrt{x^2 + y^2 - 1}\right) = \pm\left(\frac{-x\mathbf{i} - y\mathbf{j}}{\sqrt{x^2 + y^2 - 1}} + \mathbf{k}\right).$$

Since $\mathbf{n}$ forms an obtuse angle γ with the z-axis (Fig. 22), we take the minus sign and, hence,

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2 - 1}} - \mathbf{k}.$$

We find the scalar product

$$(\mathbf{a}, \mathbf{n}) = \frac{x^2 + y^2}{\sqrt{x^2 + y^2 - 1}} - \sqrt{x^2 + y^2 - 1} = \frac{1}{\sqrt{x^2 + y^2 - 1}}.$$

Using formula (7), we obtain

$$\Pi = \iint\limits_S (\mathbf{a}, \mathbf{n}^0)\, dS = \iint\limits_{D_{xy}} \frac{dx\, dy}{\sqrt{x^2+y^2-1}}.$$

Passing to the polar coordinates $x = \rho \cos \varphi$ and $y = \rho \sin \varphi$, we have

$$\Pi = \iint\limits_{D_{xy}} \frac{\rho\, d\rho\, d\varphi}{\sqrt{\rho^2-1}} = \int\limits_0^{2\pi} d\varphi \int\limits_1^2 \frac{\rho\, d\rho}{\sqrt{\rho^2-1}}$$

$$= 2\pi \sqrt{\rho^2-1}\,\Big|_1^2 = 2\sqrt{3}\,\pi.$$

Example 9. Compute the flux of the vector field

$$\mathbf{a} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$

through a closed surface bounded by the cylinder $x^2 + y^2 = R^2$ and the planes $z = x$, $z = 0$ $(z \geqslant 0)$.

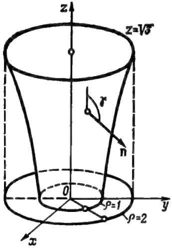

Fig. 22

Solution. The surface S is piecewise smooth and so we take advantage of the additivity property of a flux representing the desired flux Π as a sum of fluxes Π_1, Π_2, Π_3 through the smooth portions S_1 (semicircle $x^2 + y^2 \leqslant R^2$, $0 \leqslant x \leqslant R$, $z = 0$), S_2 (part of the plane $z = x$), and

S_3 (part of the cylinder $x^2 + y^2 = R^2$): $\Pi = \Pi_1 + \Pi_2 + \Pi_3$. Since S is closed, we take the outer normal to it (Fig. 23).

(1) On S_1, where $z = 0$, we have $\mathbf{n}^0 = -\mathbf{k}$ and so

$$(\mathbf{a}, \mathbf{n}^0) = -x.$$

This means that the flux

$$\Pi_1 = -\iint_{S_1} x\,dS = -\iint_{S_1} x\,dx\,dy.$$

Passing to the polar coordinates $x = \rho\cos\varphi$ and $y =$

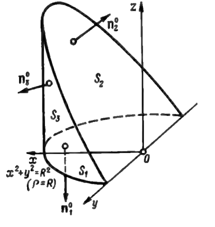

Fig. 23

$= \rho\sin\varphi$, we find

$$\Pi_1 = -\iint_{S_1} \rho\cos\varphi\rho\,d\rho\,d\varphi$$

$$= -\int_{-\pi/2}^{\pi/2} \cos\varphi\,d\varphi \int_0^R \rho^2\,d\rho = -\frac{2}{3}R^3.$$

(2) On S_2, where $z = x$, we have

$$\mathbf{n} = \pm\operatorname{grad}(z - x) = \pm(-\mathbf{i} + \mathbf{k})$$

and since the normal $\mathbf{n}$ to S_2 forms an acute angle with the z-axis, we take the plus sign in the right-hand member. Thus, $\mathbf{n} = -\mathbf{i} + \mathbf{k}$ and, hence, $(\mathbf{a}, \mathbf{n}) = x - y$.

Projecting S_2 onto the xy-plane, we get the semicircle

$$D_{xy}:\ 0 \leqslant x \leqslant \sqrt{R^2 - y^2}.$$

Then by (6) we have

$$\Pi_2 = \iint\limits_{D_{xy}} (\mathbf{a}, \mathbf{n})\Big|_{z=x} dx\, dy,$$

and again passing to polar coordinates, we find

$$\Pi_2 = \int\limits_{-\pi/2}^{\pi/2} (\cos\varphi - \sin\varphi)\, d\varphi \int\limits_0^R \rho^2\, d\rho = \frac{2}{3} R^3.$$

(3) On S_3, where $x^2 + y^2 = R^2$, that is, on the lateral surface of the cylinder, we have

$$\mathbf{n}^0 = \pm \frac{\operatorname{grad}(x^2 + y^2 - R^2)}{|\operatorname{grad}(x^2 + y^2 - R^2)|} = \pm \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}} = \pm \frac{x\mathbf{i} + y\mathbf{j}}{R}.$$

In this case, the normal $\mathbf{n}$ forms a right angle with the z-axis and therefore $\cos\gamma = 0$ and so the choice of sign in the right-hand member is arbitrary. Take the plus sign and then

$$\mathbf{n}^0 = \frac{x\mathbf{i} + y\mathbf{j}}{R}, \quad (\mathbf{a}, \mathbf{n}^0) = \frac{(x + z)\, y}{R}$$

and so

$$\Pi_3 = \frac{1}{R} \iint\limits_{S_3} (x + z)\, y\, dS.$$

It is impossible to project the surface S_3 (right cylinder) onto the xy-plane since it projects into a line, a semicircle (the one-to-one nature of the projection will be upset). The same occurs when projecting onto the xz-plane. We therefore project the surface S_3 onto the yz-plane, onto which it is projected one-to-one into the region D_{yz} bounded by the line

$$\left.\begin{aligned} x^2 + y^2 &= R^2, \\ z &= x. \end{aligned}\right\}$$

Eliminating x from this system, we obtain the equation for the projection of this line onto the zy-plane: $z^2 + y^2 =$

$= R^2$ is a circle. Since

$$|\cos\alpha| = |\cos(\widehat{\mathbf{n}_3^0, \mathbf{i}})| = |(\mathbf{n}_3^0, \mathbf{i})| = \left|\frac{x}{R}\right| = \frac{x}{R} \qquad (x \geqslant 0),$$

we will have

$$\Pi_3 = \frac{1}{R} \iint\limits_{D_{yz}} \frac{(x+z)\,y}{|\cos\alpha|}\bigg|_{x=z} dy\,dz = \iint\limits_{D_{yz}} \frac{(x+z)\,y}{x}\bigg|_{x=z} dy\,dz$$

$$= \iint\limits_{D_{yz}} \frac{2zy}{z}\,dy\,dz = 2 \iint\limits_{D_{yz}} y\,dy\,dz.$$

Using the polar coordinates $y = \rho\cos\varphi$ and $z = \rho\sin\varphi$, we get

$$\Pi_3 = 2 \iint\limits_{D_{yz}} \rho\cos\varphi\,\rho\,d\rho\,d\varphi = 2\int_0^{\pi} \cos\varphi\,d\varphi \int_0^{R} \rho^2\,d\rho = 0.$$

Thus,

$$\Pi = -\frac{2}{3}\pi R^3 + \frac{2}{3}\pi R^3 + 0 = 0.$$

108. Compute the flux of the vector field $\mathbf{a} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ through the upper side of a triangle bounded by the planes

$$x + y + z = a, \quad x = 0, \quad y = 0, \quad z = 0.$$

109. Compute the flux of the vector field $\mathbf{a} = xz\mathbf{i}$ through the outer side of the paraboloid $z = 1 - x^2 - y^2$ bounded by the plane $z = 0$ $(z \geqslant 0)$.

110. Compute the flux of the vector field $\mathbf{a} = x\mathbf{i} + z\mathbf{k}$ through the lateral surface of the circular cylinder $y = \sqrt{R^2 - x^2}$ bounded by the planes $z = 0$, $z = h$ $(h > 0)$.

111. Compute the flux of the vector field $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through the upper side of a circle cut out of the plane $z = h$ $(h > 0)$ by the cone $z = \sqrt{x^2 + y^2}$.

112. Compute the flux of the vector field $\mathbf{a} = 3x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$ through the outer side of the paraboloid $x^2 + y^2 = 9 - z$ located in the first octant.

113. Compute the flux of the vector field

$$\mathbf{a} = (x^2 + y^2)\,\mathbf{i} + (y^2 + z^2)\,\mathbf{j} + (z^2 + x^2)\,\mathbf{k}$$

through the portion of the plane $z = 0$ bounded by the circle $x^2 + y^2 = 1$ in the direction of the unit vector $\mathbf{k}$.

114. Compute the flux of the vector field $\mathbf{a} = yz\mathbf{i} - x\mathbf{j} - y\mathbf{k}$ through the total surface of the cone $x^2 + y^2 = z^2$ bounded by the plane $z = 1$ $(0 \leqslant z \leqslant 1)$.

115. Compute the flux of the vector field $\mathbf{a} = 2x\mathbf{i} + (1 - 2y)\mathbf{j} + 2z\mathbf{k}$ through the closed surface bounded by the paraboloid $x^2 + z^2 = 1 - 2y$ $(y \geqslant 0)$ and the plane $z = 0$ $(z \geqslant 0)$.

116. Compute the flux of the vector field $\mathbf{a} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ through the total surface of a pyramid bounded by the planes $x + y + z = 1$, $x = 0$, $y = 0$, $z = 0$.

117. Compute the flux of the vector field $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through the sphere $x^2 + y^2 + z^2 = R^2$.

2°. *The method of projection onto three coordinate planes.* Suppose a surface S is projected one-to-one onto all three coordinate planes. Denote by D_{xy}, D_{xz}, D_{yz} the projections of S onto the xy-, xz-, yz-planes respectively.

In that case the equation $F(x, y, z) = 0$ of the surface S is uniquely solvable for each of the arguments x, y, z so that

$$x = x(y, z), \quad y = y(x, z), \quad z = z(x, y).$$

Then the flux of the vector

$$\mathbf{a} = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

through the surface S, the unit vector of the normal of which is

$$\mathbf{n}^0 = \cos\alpha\cdot\mathbf{i} + \cos\beta\cdot\mathbf{j} + \cos\gamma\cdot\mathbf{k},$$

can be written thus:

$$\Pi = \iint_S (\mathbf{a}, \mathbf{n}^0)\,dS$$

$$= \iint_S [P(x, y, z)\cos\alpha + Q(x, y, z)\cos\beta + R(x, y, z)\cos\gamma]\,dS. \quad (8)$$

We know that

$$dS \cos \alpha = \pm dy\, dz,$$
$$dS \cos \beta = \pm dx\, dz, \qquad (9)$$
$$dS \cos \gamma = \pm dx\, dy,$$

the sign in each of the formulas of (9) being chosen to coincide with the sign of $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ on the surface S. Substituting (9) into (8), we get

$$\Pi = \pm \iint_{D_{yz}} P[x(y, z), y, z]\, dy\, dz \pm \iint_{D_{xz}} Q[x, y(x, z), z]\, dx\, dz \pm \iint_{D_{xy}} R[x, y, z(x, y)]\, dx\, dy. \qquad (10)$$

Example 10. Find the flux of the vector

$$\mathbf{a} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$$

through the portion of the external side of the sphere $x^2 + y^2 + z^2 = 1$ located in the first octant.

Solution. We have

$$\mathbf{n}^0 = \frac{\operatorname{grad}(x^2+y^2+z^2-1)}{|\operatorname{grad}(x^2+y^2+z^2-1)|} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k},$$

whence, taking into account that the surface S lies in the first octant, we obtain

$$\cos \alpha = x \geqslant 0, \quad \cos \beta = y \geqslant 0, \quad \cos \gamma = z \geqslant 0.$$

We therefore take the plus sign in (10) in front of the integrals, and putting

$$P = xy, \quad Q = yz, \quad R = xz,$$

we obtain

$$\Pi = \iint_{D_{yz}} xy\, dy\, dz + \iint_{D_{xz}} yz\, dx\, dz + \iint_{D_{xy}} xz\, dx\, dy. \qquad (11)$$

From the equation of the sphere $x^2 + y^2 + z^2 = 1$ we get

$$z = z(x, y) = \sqrt{1-x^2-y^2}, \quad y = y(x, z) = \sqrt{1-x^2-z^2}, \quad x = x(y, z) = \sqrt{1-y^2-z^2}.$$

Substituting these expressions for x, y, z respectively into the third, second, and first integrals on the right of

(11), we get

$$\Pi = \iint\limits_{D_{xy}} x\sqrt{1-x^2-y^2}\,dx\,dy + \iint\limits_{D_{xz}} z\sqrt{1-x^2-z^2}\,dx\,dz + \iint\limits_{D_{yz}} y\sqrt{1-y^2-z^2}\,dy\,dz. \quad (12)$$

Let us compute the first integral on the right and pass to the polar coordinates $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$, where $0 \leqslant \varphi \leqslant \pi/2$, $0 \leqslant \rho \leqslant 1$. This yields

$$I_1 = \iint\limits_{D_{xy}} x\sqrt{1-x^2-y^2}\,dx\,dy = \iint\limits_{D_{xy}} \rho^2\sqrt{1-\rho^2}\cos\varphi\,d\varphi\,d\rho$$

$$= \int\limits_0^{\pi/2}\cos\varphi\,d\varphi\int\limits_0^1 \rho^2\sqrt{1-\rho^2}\,d\rho = \int\limits_0^1 \rho^2\sqrt{1-\rho^2}\,d\rho.$$

Setting $\rho = \sin t$, $d\rho = \cos t\,dt$ in the last integral, we have

$$I_1 = \int\limits_0^{\pi/2}\sin^2 t\cos^2 t\,dt = \frac{1}{4}\int\limits_0^{\pi/2}\sin^2 2t\,dt = \frac{\pi}{16}.$$

The second and third integrals in (12) are computed in similar fashion and we obtain

$$I_2 = \iint\limits_{D_{xz}} z\sqrt{1-x^2-z^2}\,dx\,dz = \frac{\pi}{16},$$

$$I_3 = \iint\limits_{D_{yz}} y\sqrt{1-y^2-z^2}\,dy\,dz = \frac{\pi}{16}.$$

The desired flux is

$$\Pi = I_1 + I_2 + I_3 = \frac{3\pi}{16}.$$

118. Use the method of projecting onto all three coordinate planes to compute the flux of the vector field through a surface S.

(a) $\mathbf{a} = z\mathbf{i} - x\mathbf{j} + y\mathbf{k}$;

S is the upper side of a bounded portion of the plane $3x + 6y - 2z = 6$ cut out by the coordinate planes.

(b) $\mathbf{a} = (x + y + z)\,\mathbf{i} + (x + y + z - 1)\,\mathbf{j} - 2\mathbf{k}$;

S is the upper side of part of the plane $x + y + z - 1 = 0$ lying in the first octant.

(c) $\mathbf{a} = (x - \sqrt{y - z^2})\,\mathbf{i} + \mathbf{j} + (\sqrt{y - x^2} - z)\,\mathbf{k}$;

S is the outer side of the paraboloid of revolution $y = x^2 + z^2$ bounded by the plane $y = 4$ and lying in the first octant.

3°. *The method of introducing curvilinear coordinates on a surface.* In certain cases, when calculating the flux of a vector field through a given surface S it is possible to choose a simple coordinate system on the surface itself to compute the flux instead of projecting onto coordinate planes.

Let us consider some special cases.

Case (1). Suppose a surface S is part of the circular cylinder $x^2 + y^2 = R^2$ bounded by the surfaces $z = f_1(x, y)$ and $z = f_2(x, y)$, and we have $f_1(x, y) \leqslant f_2(x, y)$.

Setting

$$x = R\cos\varphi, \quad y = R\sin\varphi, \quad z = z,$$

we have for the given surface

$$0 \leqslant \varphi \leqslant 2\pi, \quad f_1(R\cos\varphi, R\sin\varphi) \leqslant z \leqslant f_2(R\cos\varphi, R\sin\varphi),$$

and for the element of area dS we obtain the following expression (Fig. 24):

$$dS = R\,d\varphi\,dz.$$

Then the flux of the vector field $\mathbf{a}$ through the outer side of the surface S is computed from the formula

$$\Pi = R\int_0^{2\pi} d\varphi \int_{f_1(R\cos\varphi,\ R\sin\varphi)}^{f_2(R\cos\varphi,\ R\sin\varphi)} (\mathbf{a}, \mathbf{n}^0)\,dz, \qquad (13)$$

where

$$\mathbf{n}^0 = \frac{\operatorname{grad}(x^2 + y^2 - R^2)}{|\operatorname{grad}(x^2 + y^2 - R^2)|} = \frac{x\mathbf{i} + y\mathbf{j}}{R}.$$

Example 11. Find the flux of the vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

through the outer side of the lateral surface of the circular cylinder $x^2 + y^2 = R^2$ bounded by the planes $z = 0$ and $z = H$ $(H > 0)$.

Solution. Here we have

$$0 \leqslant \varphi \leqslant 2\pi; \quad f_1(R\cos\varphi,\ R\sin\varphi) = 0$$
$$f_2(R\cos\varphi,\ R\sin\varphi) = H.$$

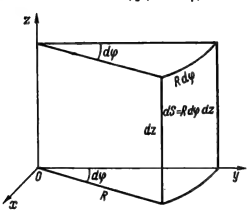

Fig. 24

Introducing curvilinear coordinates on the cylinder, we get

$$x = R\cos\varphi, \quad y = R\sin\varphi, \quad z = z.$$

Then the desired flux of the vector $\mathbf{r}$ is

$$\Pi = R\int_0^{2\pi} d\varphi \int_0^H (\mathbf{r}, \mathbf{n}^0)\, dz.$$

But since

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = R\cos\varphi\cdot\mathbf{i} + R\sin\varphi\cdot\mathbf{j} + z\mathbf{k}$$

and the normal $\mathbf{n}^0$ on the cylinder is

$$\mathbf{n}^0 = \frac{x\mathbf{i}+y\mathbf{j}}{R} = \frac{R\cos\varphi\cdot\mathbf{i}+R\sin\varphi\cdot\mathbf{j}}{R} = \cos\varphi\cdot\mathbf{i}+\sin\varphi\cdot\mathbf{j},$$

it follows that the scalar product on the cylinder will be

$$(\mathbf{r}, \mathbf{n}^0) = R\cos^2\varphi + R\sin^2\varphi = R.$$

Finally, we obtain

$$\Pi = R^2 \int_0^{2\pi} d\varphi \int_0^H dz = 2\pi R^2 H.$$

Example 12. Compute the flux of the radius vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

through the lateral surface of the circular cylinder $x^2 + y^2 = 1$ bounded from below by the plane $x + y + z = 1$ and from above by the plane $x + y + z = 2$.

Solution. Here (Fig. 25) we have

$R = 1, f_1(x, y) = 1 - x - y, f_2(x, y) = 2 - x - y.$

Passing to coordinates on the cylinder

$x = \cos\varphi, \quad y = \sin\varphi,$

$z = z,$

we get

$f_1(x, y) = 1 - \cos\varphi - \sin\varphi,$

$f_2(x, y) = 2 - \cos\varphi - \sin\varphi.$

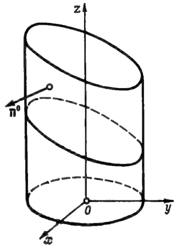

Fig. 25

According to (13), the flux of the vector $\mathbf{r}$ is

$$\Pi = \int_0^{2\pi} d\varphi \int_{1-\cos\varphi-\sin\varphi}^{2-\cos\varphi-\sin\varphi} (\mathbf{r}, \mathbf{n}^0)\, dz.$$

But since on the cylinder $x^2 + y^2 = 1$ we have

$$\mathbf{n}^0 = x\mathbf{i} + y\mathbf{j} = \cos\varphi \cdot \mathbf{i} + \sin\varphi \cdot \mathbf{j},$$

it follows that

$$(\mathbf{r}, \mathbf{n}^0) = x^2 + y^2 = \cos^2\varphi + \sin^2\varphi = 1$$

and, hence,

$$\Pi = \int_0^{2\pi} d\varphi \int_{1-\cos\varphi-\sin\varphi}^{2-\cos\varphi-\sin\varphi} dz = \int_0^{2\pi} d\varphi = 2\pi.$$

119. Find the flux of the vector

$$\mathbf{a} = y\mathbf{i} + x\mathbf{j} - e^{xyz}\mathbf{k}$$

through the outer side of the lateral surface of the cylinder $x^2 + y^2 = 4$ bounded by the planes $z = 0$ and $x + y + z = 4$.

120. Find the flux of the vector

$$\mathbf{a} = x\mathbf{i} - xy\mathbf{j} + z\mathbf{k}$$

through the outer side of the cylindrical surface $x^2 + z^2 = R^2$ bounded by the planes $y = 1$ and $x + y = 4$.

121. Find the flux of the vector

$$\mathbf{a} = x^3\mathbf{i} - y^3\mathbf{j} + xz^3\mathbf{k}$$

through the outer side of the cylindrical surface $x^2 + y^2 = 9$ bounded by the sphere $x^2 + y^2 + z^2 = 25$.

122. Find the flux of the vector field

$$\mathbf{a} = x\mathbf{i} - y\mathbf{j} - xyz^3\mathbf{k}$$

through the outer side of the lateral surface of the cylinder $x^2 + y^2 = 1$ bounded by the plane $z = 0$ and the hyperbolic paraboloid $z = x^2 - y^2$.

123. Find the flux of the vector field

$$\mathbf{a} = (xy - y^2)\,\mathbf{i} + (2x - x^2 + xy)\,\mathbf{j} + z\mathbf{k}$$

through the outer side of the lateral surface of the cylinder $x^2 + y^2 = 1$ bounded by the elliptic cone $z^2 = x^2/2 + y^2$.

Case (2). Suppose the surface S is a part of the sphere $x^2 + y^2 + z^2 = R^2$ bounded by conical surfaces whose equations in spherical coordinates have the form $\theta = f_1(\varphi)$, $\theta = f_2(\varphi)$ and by the half-planes $\varphi = \varphi_1$, $\varphi = \varphi_2$.

For the points of the given sphere, set

$$x = R\cos\varphi\sin\theta, \qquad y = R\sin\varphi\sin\theta, \qquad z = R\cos\theta,$$

where $\varphi_1 \leqslant \varphi \leqslant \varphi_2$ and $\theta_1 \leqslant \theta \leqslant \theta_2$. Then for the element of area dS we obtain (Fig. 26)

$$dS = R^2 \sin\theta \, d\theta \, d\varphi.$$

In this case, the flux of the vector field **a** through the outer part S of the sphere is found from the formula

$$\Pi = R^2 \int_{\varphi_1}^{\varphi_2} d\varphi \int_{\theta_1}^{\theta_2} (\mathbf{a}, \mathbf{n}^0) \sin\theta \, d\theta, \tag{14}$$

where

$$\mathbf{n}^0 = \frac{\operatorname{grad}(x^2+y^2+z^2-R^2)}{|\operatorname{grad}(x^2+y^2+z^2-R^2)|} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{R}.$$

Example 13. Find the flux of the vector

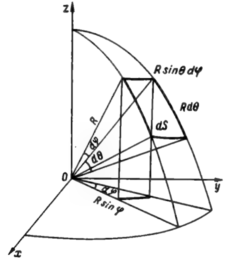

Fig. 26

$$\mathbf{a} = (x - 2y + 1)\,\mathbf{i} + (2x + y - 3z)\,\mathbf{j} + (3y + z)\,\mathbf{k}$$

through a part of the surface of the sphere $x^2 + y^2 + z^2 = 1$ located in the first octant into a region where $x^2 + y^2 + z^2 > 1$.

Solution. Here we have

$$R = 1,\ \varphi_1 = 0,\ \varphi_2 = \pi/2,\ \theta_1 = 0,\ \theta_2 = \pi/2,$$

$$\mathbf{n}^0 = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},\ (\mathbf{a},\ \mathbf{n}^0) = x^2 + y^2 + z^2 + x.$$

On the sphere $x^2 + y^2 + z^2 = 1$ we introduce coordinates φ and θ so that

$$x = \cos\varphi \sin\theta, \quad y = \sin\varphi \sin\theta, \quad z = \cos\theta.$$

We then have

$$(\mathbf{a},\ \mathbf{n}^0) = 1 + \cos\varphi \sin 0$$

and, using (14), we obtain

$$\Pi = \int_0^{\pi/2} d\varphi \int_0^{\pi/2} (1 + \cos\varphi \sin\theta) \sin\theta \, d\theta$$

$$= \int_0^{\pi/2} d\varphi \int_0^{\pi/2} \sin\theta \, d\theta + \int_0^{\pi/2} \cos\varphi \, d\varphi \int_0^{\pi/2} \sin^2\theta \, d\theta = \frac{3}{4}\pi.$$

124. Find the flux of the vector field

$$\mathbf{a} = x^3\mathbf{i} - y^3\mathbf{j} + z\mathbf{k}$$

through the outer side of that part of the sphere $x^2 + y^2 + z^2 = 1$ cut out by the conical surface $z^2 = x^2 + y^2$, $z \geqslant \sqrt{x^2 + y^2}$.

125. Find the flux of the vector field

$$\mathbf{a} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

through the outer side of that part of the sphere $x^2 + y^2 + z^2 = R^2$ located in the first octant.

126. Find the flux of the radius vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

through the outer side of that part of the sphere $x^2 + y^2 + z^2 = 2$ that is bounded by the planes $z = 0$ and $z = y$.

127. Find the flux of the vector

$$\mathbf{a} = xz\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}$$

through the outer part of the sphere $x^2 + y^2 + z^2 = 9$ cut off by the plane $z = 2$ $(z \geqslant 2)$.

Sec. 12. The flux of a vector through a closed surface. The Gauss-Ostrogradsky theorem

Theorem. *If in some region G of space the coordinates of a vector*

$$\mathbf{a} = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

are continuous and have continuous partial derivatives $\partial P/\partial x$, $\partial Q/\partial y$, $\partial R/\partial z$, *then the flux of the vector* **a** *through any closed piecewise smooth surface* Σ *located in G is equal to the triple integral of* $\partial P/\partial x + \partial Q/\partial y + \partial R/\partial z$ *over the region V bounded by the surface* Σ:

$$\Pi = \oint\!\!\oint_{\Sigma} (\mathbf{a}, \mathbf{n}^0)\, d\sigma = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dv \qquad (1)$$

(the *Gauss-Ostrogradsky formula*).

The normal n to the surface Σ is taken to be the outer normal.

Example 1. Compute the flux of the vector

$$\mathbf{a} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$$

through the closed surface

$$x^2 + y^2 + z^2 = R^2, \quad z = 0 \ (z > 0).$$

Solution. By formula (1),

$$\Pi = \iiint_V (2x + 2y + 2z)\, dv. \qquad (2)$$

The integral (2) is conveniently computed in the spherical coordinates r, θ, φ. We have

$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r\cos\theta$$

and the element of volume is

$$dv = r^2 \sin\theta \, dr\, d\theta\, d\varphi$$

so that

$$\Pi = 2 \iiint_V (r \sin\theta\cos\varphi + r\sin\theta\sin\varphi$$

$$+ r\cos\theta)\, r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= 2\int_0^{2\pi} d\varphi \int_0^{\pi/2} \sin\theta\,(\sin\theta\cos\varphi + \sin\theta\sin\varphi + \cos\theta)\,d\theta \int_0^R r^3\,dr$$

$$= \frac{2R^4}{3}\int_0^{2\pi} d\varphi \int_0^{\pi/2} \cos\theta\sin\theta\,d\theta = \frac{\pi R^4}{2}.$$

Example 2. Compute the flux of the vector

$$\mathbf{a} = 4x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$$

through the surface of a torus.

Solution. Taking advantage of the Gauss-Ostrogradsky theorem we find that the desired flux is

$$\Pi = \oint\!\!\!\oint_S (\mathbf{a}, \mathbf{n}^0)\,d\sigma = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dv = 4V,$$

where V is the volume of the torus. To compute the volume V, let us take advantage of the Guldin theorem

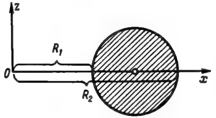

Fig. 27

on the volume of a solid of revolution, by virtue of which the volume is equal to the product of the area of the rotating figure into the path covered by the centre of mass of the figure during the rotation.

Let R_1 and R_2 be the inner and outer radii of the torus (Fig. 27). The area S of the circle, which during rotation forms the torus, is equal to

$$S = \pi\left(\frac{R_2 - R_1}{2}\right)^2.$$

The path length described by the centre of mass (the centre of that circle) is the circumference l of a circle of radius $(R_1 + R_2)/2$, that is,

$$l = 2\pi \frac{R_1+R_2}{2} = \pi (R_1 + R_2).$$

Thus, the volume V of the torus is equal to

$$V = \pi \left(\frac{R_2 - R_1}{2}\right)^2 \pi (R_2 + R_1) = \frac{\pi^2}{4} (R_2 - R_1)^2 (R_2 + R_1).$$

The desired flux is

$$\Pi = \pi^2 (R_2 - R_1)^2 (R_2 + R_1).$$

Example 3. Using the Gauss-Ostrogradsky theorem, compute the flux of the vector field

$$\mathbf{a} = \left(\frac{x^2 y}{1+y^2} + 6yz^2\right)\mathbf{i} + 2x \arctan y \cdot \mathbf{j} - \frac{2xz\,(1+y) + 1 + y^2}{1+y^2}\,\mathbf{k}$$

through the outer side of that part of the surface $z = 1 - x^2 - y^2$ located above the xy-plane.

Solution. In order to be able to apply the Gauss-Ostrogradsky theorem, close the given surface from below with a portion of the xy-plane that is bounded by the circle

$$\left.\begin{aligned} x^2 + y^2 &= 1, \\ z &= 0. \end{aligned}\right\}$$

Let V be the volume of the resulting solid bounded by a closed piecewise smooth surface σ consisting of a part σ_1 of the paraboloid of revolution $z = 1 - x^2 - y^2$ and a part σ_2 of the plane $z = 0$ (Fig. 28).

The flux of the given vector through the surface σ is, by the Gauss-Ostrogradsky theorem, equal to

$$\Pi = \oiint_{\sigma} (\mathbf{a}, \mathbf{n}^0)\, d\sigma = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dv.$$

We find the sum

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{2xy}{1+y^2} + \frac{2x}{1+y^2} - \frac{2x\,(1+y)}{1+y^2} \equiv 0.$$

Consequently, the flux is

$$\Pi = \oiint_{\sigma} (\mathbf{a}, \mathbf{n}^0)\, d\sigma = 0.$$

By virtue of the additivity of the flux we have

$$\Pi = \iint_{\sigma_1} (\mathbf{a}, \mathbf{n}^0)\, d\sigma + \iint_{\sigma_2} (\mathbf{a}, \mathbf{n}^0)\, d\sigma = 0.$$

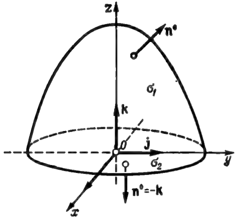

Fig. 28

From this the desired flux is

$$\Pi_1 = \iint_{\sigma_1} (\mathbf{a}, \mathbf{n}^0)\, d\sigma = -\iint_{\sigma_2} (\mathbf{a}, \mathbf{n}^0)\, d\sigma.$$

The flux Π_2 of the vector $\mathbf{a}$ through the circle $x^2 + y^2 \leqslant 1$, $z = 0$ is equal to

$$\Pi_2 = \iint_{\sigma_2} (\mathbf{a}, \mathbf{n}^0)\, d\sigma.$$

Since on the plane $z = 0$, we have

$$\mathbf{a} = \frac{x^2 y}{1+y^2}\,\mathbf{i} + 2x \arctan y \cdot \mathbf{j} - \mathbf{k}, \quad \mathbf{n}^0 = -\mathbf{k},$$

and hence $(\mathbf{a}, \mathbf{n}^0) = 1$, it follows that the flux Π_2 through the circle σ_2 is equal to the area of the circle σ_2:

$$\Pi_2 = \iint_{\sigma_2} d\sigma = \pi.$$

The desired flux $\Pi_1 = -\Pi_2 = -\pi$.

By appropriately closing the given unclosed surfaces and making use of the Gauss-Ostrogradsky theorem, compute the fluxes of the vector fields through the indicated surfaces (the normal to the closed surface is the outer normal).

128. $\mathbf{a} = (1 - 2x)\,\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; S: $x^2 + y^2 = z^2$ $(0 \leqslant z \leqslant 4)$.

129. $\mathbf{a} = z^2\mathbf{i} + xz\mathbf{j} + y\mathbf{k}$; S: $x^2 + y^2 = 4 - z$ $(z \geqslant 0)$.

130. $\mathbf{a} = (y^2 + z^2)\,\mathbf{i} - y^2\mathbf{j} + 2yz\mathbf{k}$; S: $x^2 + z^2 = y^2$ $(0 \leqslant y \leqslant 1)$.

Sec. 13. The divergence of a vector field. Solenoidal fields

The notion of the flux of a vector through a closed surface leads to the concept of the divergence of a field. This concept yields a certain quantitative characteristic of a field at each point in the field.

Let M be a point of the field under study. Surround it by a surface Σ of arbitrary shape, for instance, a sphere of sufficiently small radius. Let the region bounded by the surface Σ be (V) and its volume V. We consider the ratio

$$\frac{\oiint\limits_{\Sigma} (\mathbf{a}, \mathbf{n}^0)\, d\sigma}{V}. \tag{1}$$

Definition. If the ratio (1) has a finite limit when the region (V) contracts to the point M, then this limit is termed the *divergence* of the vector field (the divergence of the vector $\mathbf{a}$) at the point M and is designated as

div **a** (M). We thus have

$$\operatorname{div} \mathbf{a}(M) = \lim_{(V) \to M} \frac{\oint\!\!\oint_{\Sigma} (\mathbf{a}, \mathbf{n}^0)\, d\sigma}{V}. \tag{2}$$

Formula (2) yields an invariant definition of divergence. This definition means that the divergence of the field **a** at the point M constitutes the volume density of the flux of the vector **a** at that point.

The points M of the vector field **a** (M) at which div **a** > 0 are termed *sources*, while the points at which div **a** < 0 are termed *sinks* of the vector field.

The divergence of a vector field is a scalar function of the points of the field.

If the coordinates of the vector

$$\mathbf{a}(M) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

have continuous partial derivatives $\partial P/\partial x$, $\partial Q/\partial y$, $\partial R/\partial z$ in the neighbourhood of the point $M(x, y, z)$, then, using the invariant definition of divergence, we find from the Gauss-Ostrogradsky theorem that

$$\operatorname{div} \mathbf{a} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \tag{3}$$

All the quantities in (3) are considered at the same point $M(x, y, z)$.

Using (3) for divergence, we can write the Gauss-Ostrogradsky theorem (see Sec. 12) in vector form:

$$\oint\!\!\oint_{\Sigma} (\mathbf{a}, \mathbf{n}^0)\, d\sigma = \iiint_{V} \operatorname{div} \mathbf{a}\, dv \tag{4}$$

Example 1. Using the invariant definition, compute the divergence of the vector $\mathbf{a} = x\mathbf{i}$ at the point $O(0, 0, 0)$ taking for the surfaces σ surrounding the point O the spheres σ_ε of radius ε centred at that point.

Solution. By the definition of divergence, we have at the given point

$$\operatorname{div} \mathbf{a}(0) = \lim_{(\sigma_\varepsilon)\to 0} \frac{\oiint\limits_{\sigma_\varepsilon} (\mathbf{a}, \mathbf{n}^0)\, d\sigma}{v_\varepsilon},$$

where v_ε is the volume of a ball bounded by a sphere σ_ε, or

$$\operatorname{div} \mathbf{a}(0) = \lim_{\varepsilon\to 0} \frac{\oiint\limits_{\sigma_\varepsilon} (\mathbf{a}, \mathbf{n}^0)\, d\sigma}{v_\varepsilon}.$$

But since the volume of the ball is $v_\varepsilon = 4\pi\varepsilon^3/3$, it follows that

$$\operatorname{div} \mathbf{a}(0) = \lim_{\varepsilon\to 0} \frac{\oiint\limits_{\sigma_\varepsilon} (\mathbf{a}, \mathbf{n}^0)\, d\sigma}{\frac{4}{3}\pi\varepsilon^3}.$$

Let us compute the flux $\oiint\limits_{\sigma_\varepsilon} (\mathbf{a}, \mathbf{n}^0)\, d\sigma$ of the given vector through the sphere σ_ε. The unit vector of the normal $\mathbf{n}^0$ to the sphere σ_ε is directed along a radius of the sphere, and so we can put

$$\mathbf{n}^0 = \mathbf{r}^0 = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\mathbf{r}}{\varepsilon},$$

where $\mathbf{r}^0$ is the unit vector of the radius vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, or

$$\mathbf{n}^0 = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\varepsilon}.$$

The desired flux is

$$\oiint\limits_{\sigma_\varepsilon} (\mathbf{a}, \mathbf{n}^0)\, d\sigma = \oiint\limits_{\sigma_\varepsilon} \frac{x^2}{\varepsilon}\, d\sigma.$$

Passing to coordinates on the sphere σ_ε,
$x = \varepsilon \cos\varphi \sin\theta$, $y = \varepsilon \sin\varphi \sin\theta$, $z = \varepsilon \cos\theta$,
we get

$$\oiint_{\sigma_\varepsilon} (\mathbf{a}, \mathbf{n}^0)\, d\sigma = \iint_{\sigma_\varepsilon} \frac{\varepsilon^2 \cos^2\varphi \sin^2\theta\, \varepsilon^2 \sin\theta\, d\varphi\, d\theta}{\varepsilon}$$

$$= \varepsilon^3 \int_0^{2\pi} \cos^2\varphi\, d\varphi \int_0^{\pi} \sin^3\varphi\, d\varphi = \frac{4}{3}\pi\varepsilon^3.$$

Consequently

$$\operatorname{div} \mathbf{a}(0) = \lim_{\varepsilon\to 0} \frac{\frac{4}{3}\pi\varepsilon^3}{\frac{4}{3}\pi\varepsilon^3} = 1.$$

Example 2. Compute div **r**.

Solution. We have $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, so that $P = x$, $Q = y$, $R = z$ and, hence, by formula (3),

$$\operatorname{div} \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

Example 3. Compute div $(u \cdot \mathbf{a})$, where $u(M)$ is a scalar function and $\mathbf{a}(M) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ is a vector function.

Solution. Using formula (3), we get

$$\operatorname{div}(u\mathbf{a}) = \frac{\partial(uP)}{\partial x} + \frac{\partial(uQ)}{\partial y} + \frac{\partial(uR)}{\partial z} = u\frac{\partial P}{\partial x} + P\frac{\partial u}{\partial x} + u\frac{\partial Q}{\partial y} + Q\frac{\partial u}{\partial y}$$
$$+ u\frac{\partial R}{\partial z} + R\frac{\partial u}{\partial z} = u\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) + P\frac{\partial u}{\partial x}$$
$$+ Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = u \operatorname{div} \mathbf{a} + (\mathbf{a}, \operatorname{grad} u).$$

Thus

$$\operatorname{div}(u\mathbf{a}) = u \operatorname{div} \mathbf{a} + (\mathbf{a}, \operatorname{grad} u). \tag{5}$$

Example 4. Find the divergence of the vector

$$\mathbf{a} = \varphi(r)\,\mathbf{r}^0 = \frac{\varphi(r)}{r}\mathbf{r},$$

where $r = |\mathbf{r}|$ is the distance from the coordinate origin to the variable point $M(x, y, z)$.

Solution. Using formula (5), we obtain

$$\operatorname{div} \mathbf{a} = \frac{\varphi(r)}{r} \operatorname{div} \mathbf{r} + \left(\mathbf{r}, \operatorname{grad} \frac{\varphi(r)}{r}\right).$$

Furthermore,

$$\operatorname{div} \mathbf{r} = 3, \quad \operatorname{grad} \frac{\varphi(r)}{r} = \left(\frac{\varphi(r)}{r}\right)' \operatorname{grad} r = \frac{r\varphi'(r) - \varphi(r)}{r^2} \mathbf{r}^0.$$

And so

$$\operatorname{div} \mathbf{a} = \frac{\varphi(r)}{r} \cdot 3 + \left(\frac{r\varphi'(r) - \varphi(r)}{r^2} \mathbf{r}^0, \mathbf{r}\right)$$
$$= 3\frac{\varphi(r)}{r} + \frac{r\varphi'(r) - \varphi(r)}{r} = 2\frac{\varphi(r)}{r} + \varphi'(r).$$

131. For what function $\psi(r)$ will we have $\operatorname{div} \psi(r)\mathbf{r} = 2\psi(r)$?

132. Find $\operatorname{div}(r^4\mathbf{r})$.

133. Find the divergence of the vector field

$$\mathbf{a} = [\mathbf{c}, \mathbf{r}],$$

where $\mathbf{c}$ is a constant vector.

134. Find

$$\operatorname{div}(\mathbf{r}[\mathbf{w}, \mathbf{r}]),$$

where $\mathbf{w}$ is a constant vector.

135. For what function $\psi(z)$ will the divergence of the field

$$\mathbf{a} = xz\mathbf{i} + y\mathbf{j} + \psi(z)\mathbf{k}$$

be equal to z?

136. Find the flux of the radius vector $\mathbf{r}$ through the surface of a sphere.

137. The electrostatic field of a point charge q is

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0} \frac{\mathbf{r}^0}{r^2}.$$

Compute $\operatorname{div} \mathbf{E}$.

138. Show that

$$\frac{1}{3} \oiint_{\Sigma} (\mathbf{r}, \mathbf{n}^0)\, d\sigma = V,$$

where V is the volume bounded by a closed surface Σ.

139. Prove that if Σ is a closed piecewise smooth surface and $\mathbf{c}$ is a nonzero constant vector, then

$$\iint\limits_{\Sigma}\!\!\!\!\!\!\!\bigcirc \cos(\mathbf{n}, \mathbf{c})\, d\sigma = 0,$$

where $\mathbf{n}$ is a vector normal to the surface Σ.

140. Prove the formula

$$\iint\limits_{\Sigma}\!\!\!\!\!\!\!\bigcirc (\varphi\mathbf{a}, \mathbf{n}^0)\, d\sigma = \iiint\limits_{V} (\varphi \operatorname{div} \mathbf{a} + (\mathbf{a}, \operatorname{grad} \varphi))\, dv,$$

where $\varphi = \varphi(x, y, z)$ and Σ is a surface bounding the volume V. Establish the conditions under which this formula is applicable.

141. Prove that if the function $u(x, y, z)$ satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

then

$$\iint\limits_{\Sigma}\!\!\!\!\!\!\!\bigcirc \frac{\partial u}{\partial n}\, d\sigma = 0,$$

where $\partial u/\partial n$ is the derivative with respect to the direction of the outer normal to the piecewise smooth closed surface Σ.

142. Prove that if the function $u(x, y, z)$ is a second-degree polynomial and Σ is a piecewise smooth closed surface, then the integral

$$\iint\limits_{\Sigma}\!\!\!\!\!\!\!\bigcirc \frac{\partial u}{\partial n}\, d\sigma$$

is proportional to the volume bounded by the surface Σ.

Find the flux of a vector field through the indicated closed surfaces: (1) directly, (2) via the Gauss-Ostrogradsky theorem.

143. $\mathbf{a} = x\mathbf{i} + z\mathbf{k}$; S: $\begin{cases} z = x^2 + y^2, \\ z = 4. \end{cases}$

144. $\mathbf{a}=2x\mathbf{i}+2y\mathbf{j}-z\mathbf{k}$; S: $\begin{cases} z^2=x^2+y^2, \\ z=H, \ H>0. \end{cases}$

145. $\mathbf{a}=x\mathbf{i}-z\mathbf{j}$; S: $\begin{cases} z=6-x^2-y^2, \\ z^2=x^2+y^2, \ z\geqslant 0. \end{cases}$

146. $\mathbf{a}=yz\mathbf{i}-x\mathbf{j}-y\mathbf{k}$; S: $\begin{cases} x^2+z^2=y^2, \\ y=1 \ \ (0\leqslant y\leqslant 1). \end{cases}$

147. $\mathbf{a}=x\mathbf{i}+2y\mathbf{j}-z\mathbf{k}$; S: $\begin{cases} z^2=x^2+y^2, \\ z=x^2+y^2. \end{cases}$

148. $\mathbf{a}=2x\mathbf{i}-(z-1)\mathbf{k}$; S: $\begin{cases} x^2+y^2=4, \\ z=0, \ z=1. \end{cases}$

149. $\mathbf{a}=2x\mathbf{i}-y\mathbf{j}+z\mathbf{k}$; S: $\begin{cases} x^2+y^2+z^2=4, \\ 3z=x^2+y^2 \ \left(z\geqslant \dfrac{x^2+y^2}{3}\right). \end{cases}$

150. $\mathbf{a}=yx\mathbf{i}+2y\mathbf{j}-z\mathbf{k}$; S: $x^2+y^2+z^2=4$.

151. $\mathbf{a}=(x+z)\mathbf{i}+(y+x)\mathbf{j}+(z+x)\mathbf{k}$;

S: $\begin{cases} x^2+y^2=R^2 \\ z=y, \ z\geqslant 0. \end{cases}$

152. $\mathbf{a}=3x\mathbf{i}-y\mathbf{j}-z\mathbf{k}$; S: $\begin{cases} 9-z=x^2+y^2, \\ x=0, \ y=0, \ z=0 \ \ \text{(first octant).} \end{cases}$

153. $\mathbf{a}=(y-x)\mathbf{i}+(z-y)\mathbf{j}+(x-z)\mathbf{k}$; S: $\begin{cases} x+y+z=1, \\ x-y+z=1, \\ x=0, \ z=0. \end{cases}$

154. $\mathbf{a}=x\mathbf{i}-2y\mathbf{j}-z\mathbf{k}$; S: $\begin{cases} 1-z=x^2+y^2, \\ z=0. \end{cases}$

Solenoidal fields

Definition. If at all points M of a certain region G the divergence of a vector field (specified in G) is zero,

$$\operatorname{div} \mathbf{a}\,(M)=0,$$

then we say that the field is *solenoidal* in that region.

Thus, a solenoidal field is, by definition, without sources and sinks.

From the Gauss-Ostrogradsky theorem it follows that in a solenoidal field the flux of a vector $\mathbf{a} = \mathbf{a}(M)$ through any closed surface σ lying in the field is zero:

$$\oiint_{\sigma} (\mathbf{a}, \mathbf{n}^0)\, d\sigma = 0.$$

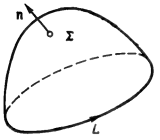

Fig. 29

In a solenoidal field G, the vector lines cannot begin or end. They can either be closed curves or have ends on the boundary of the field.

The equation

$$\operatorname{div} \mathbf{a}(M) = 0$$

is encountered in hydrodynamics and is termed the *continuity equation of an incompressible fluid.*

In this case, the amount of fluid exiting through some closed surface σ is always equal to the amount entering, and the total flux is zero.

Which of the following vector fields is a solenoidal field?

155. $\mathbf{a} = x(z^2 - y^2)\,\mathbf{i} + y(x^2 - z^2)\,\mathbf{j} + z(y^2 - x^2)\,\mathbf{k}$.

156. $\mathbf{a} = y^2\mathbf{i} - (x^2 + y^2)\,\mathbf{j} + z(3y^2 + 1)\,\mathbf{k}$.

157. $\mathbf{a} = (1 + 2xy)\,\mathbf{i} - y^2z\mathbf{j} + (z^2y - 2zy + 1)\,\mathbf{k}$.

158. Show that the field of the vector

$$\mathbf{E} = \frac{q}{r^2} r^0 \quad \left(r = \sqrt{x^2 + y^2 + z^2}\right)$$

is solenoidal throughout the region that does not contain the coordinate origin $O(0, 0, 0)$.

159. Under what condition will the vector field $\mathbf{a} = \varphi(r)\,\mathbf{r}$ be solenoidal?

Suppose we have a field $\mathbf{a}(M)$, which is not necessarily solenoidal. In the field, consider a closed oriented contour L. The surface Σ containing the curve L as its edge will be called a *surface spanned by the contour L*. Let us agree to orient the normal $\mathbf{n}$ to the surface Σ so that the chosen circuit about the contour L will be seen from the

end of the normal as being counterclockwise (Fig. 29).

160. Show that in a solenoidal field the flux of a vector $\mathbf{a}(M)$ does not depend on the type of surface Σ spanned by the given contour L and depends solely on the contour itself.

Sec. 14. A line integral in a vector field. The circulation of a vector field

Suppose we have a continuous vector field $\mathbf{a} = \mathbf{a}(M)$ and a piecewise smooth curve L on which a positive direction has been chosen (in other words, L is an oriented curve).

Definition 1. The *line integral* from the vector $\mathbf{a} = \mathbf{a}(M)$ along the oriented curve L is termed a line integral of the first kind (an integral over the arc length of a curve) of a scalar product $(\mathbf{a}, \boldsymbol{\tau}^0)$:

$$\int_L (\mathbf{a}, \boldsymbol{\tau}^0)\, ds,$$

where $\boldsymbol{\tau}^0 = \boldsymbol{\tau}^0(M)$ is the unit vector of the vector tangent to the curve L whose orientation coincides with that of L; ds is the differential of arc length s of L.

If $\mathbf{r} = \mathbf{r}(M)$ is the radius vector of an arbitrary point M of the curve L, then the line integral in the field $\mathbf{a}(M)$ may be written thus:

$$\int_L (\mathbf{a}, \boldsymbol{\tau}^0)\, ds = \int_L (\mathbf{a}, d\mathbf{r}), \tag{1}$$

If a rectangular coordinate xyz-system is introduced in the vector field, then $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$,

$$\mathbf{a} = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

and the line integral (1) can be expressed in terms of a line integral of the second kind:

$$\int_L (\mathbf{a}, d\mathbf{r}) = \int_L P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz.$$

When $\mathbf{a} = \mathbf{a}(M)$ is a force field, the line integral (1) yields the magnitude of the work of that field over the curve L.

Properties of a line integral

(a) Linearity:

$$\int_L (\lambda \mathbf{a}_1 + \mu \mathbf{a}_2, d\mathbf{r}) = \lambda \int_L (\mathbf{a}_1, d\mathbf{r}) + \int_L (\mathbf{a}_2, d\mathbf{r}),$$

where λ and μ are constants.

(b) Additivity:

$$\int_{L_1+L_2} (\mathbf{a}, d\mathbf{r}) = \int_{L_1} (\mathbf{a}, d\mathbf{r}) + \int_{L_2} (\mathbf{a}, d\mathbf{r}),$$

(c) The integral reverses sign with a change in the orientation of L:

$$\int_{BA} (\mathbf{a}, d\mathbf{r}) = -\int_{AB} (\mathbf{a}, d\mathbf{r}),$$

where A is the initial and B is the terminal point of the curve L.

Calculating a line integral in a vector field

Let the curve L be specified by the parametric equations

$$x = \varphi(t), \quad y = \psi(t), \quad z = \chi(t) \quad (t_0 \leqslant t \leqslant t_1).$$

Here, at the initial point A of L the parameter t assumes the value $t = t_0$ and at the terminal point B of L it assumes the value $t = t_1$ (the direction on L corresponds to increasing values of the parameter t from t_0 to t_1); the functions $\varphi(t)$, $\psi(t)$, $\chi(t)$ have continuous derivatives on the interval $[t_0, t_1]$. Then

$$\int_L (\mathbf{a}, d\mathbf{r}) = \int_{AB} (\mathbf{a}, d\mathbf{r}) = \int_{t_0}^{t_1} \{P[\varphi(t), \psi(t), \chi(t)]\varphi'(t)$$
$$+ Q[\varphi(t), \psi(t), \chi(t)]\psi'(t) + R[\varphi(t), \psi(t), \chi(t)]\chi'(t)\}\, dt.$$

If the curve L is given by a system of equations $y = \psi(x)$, $z = \chi(x)$, $a \leqslant x \leqslant b$, then

$$\int_{AB} (\mathbf{a}, d\mathbf{r}) = \int_a^b \{P[x, \psi(x), \chi(x)] + Q[x, \psi(x), \chi(x)]\psi'(x)$$
$$+ R[x, \psi(x), \chi(x)]\chi'(x)\}\, dx.$$

Similar formulas may also be written for cases where the line is specified by one of the following systems of equations:

$$x = \varphi(y), \quad z = \chi(y) \quad (y_0 \leqslant y \leqslant y_1)$$

or

$$x = \varphi(z), \quad y = \psi(z) \quad (z_0 \leqslant z \leqslant z_1).$$

Example 1. Find the line integral of the vector $\mathbf{a} = \mathbf{r}/|\mathbf{r}|$, where $\mathbf{r}$ is a radius vector on the line segment from point $A(\mathbf{r}_A)$ to point $B(\mathbf{r}_B)$.

Solution. The desired line integral is

$$\int_{AB} (\mathbf{a}, d\mathbf{r}) = \int_{AB} \frac{(\mathbf{r}, d\mathbf{r})}{|\mathbf{r}|}. \tag{1}$$

From

$$d(\mathbf{r}, \mathbf{r}) = (d\mathbf{r}, \mathbf{r}) + (\mathbf{r}, d\mathbf{r}) = 2(\mathbf{r}, d\mathbf{r})$$

we find

$$(\mathbf{r}, d\mathbf{r}) = \frac{1}{2} d(\mathbf{r}, \mathbf{r}) = \frac{1}{2} d(|\mathbf{r}|^2) = \frac{1}{2} \cdot 2|\mathbf{r}|\, d|\mathbf{r}| = |\mathbf{r}|\, d|\mathbf{r}|,$$

whence

$$\frac{(\mathbf{r}, d\mathbf{r})}{|\mathbf{r}|} = d|\mathbf{r}|. \tag{2}$$

Substituting (2) into the integral (1), we get

$$\int_{AB} (\mathbf{a}, d\mathbf{r}) = \int_{AB} d|\mathbf{r}| = \int_{\mathbf{r}_A}^{\mathbf{r}_B} d|\mathbf{r}| = |\mathbf{r}_B| - |\mathbf{r}_A|.$$

Note that

$$|d\mathbf{r}| \neq d|\mathbf{r}|.$$

Find the line integral over the line segment bounded by the points $A(\mathbf{r}_1)$ and $B(\mathbf{r}_2)$ for the following vector fields:

161. $\mathbf{a} = \mathbf{r}$.

162. $\mathbf{a} = \dfrac{\mathbf{r}}{|\mathbf{r}|^2}$.

163. $\mathbf{a} = \dfrac{\mathbf{r}^0}{|\mathbf{r}|^2}$, $\mathbf{r}^0$ is the unit vector.

164. Compute the line integral over the straight line passing through the points $O(0, 0, 0)$ and $M_1(1, 1, 1)$

from point O to point M_1 if $\mathbf{a} = [\mathbf{b}, \mathbf{r}]$, where $\mathbf{b}$ is a constant vector.

165. Prove that

$$\int_{AB} (\operatorname{grad} u, d\mathbf{r}) = u(B) - u(A).$$

Example 2. Find the line integral of the vector

$$\mathbf{a} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$$

over an arc L of the helical curve

$$x = R\cos t, \quad y = R\sin t, \quad z = \frac{t}{2\pi}$$

from point A, the point of intersection of the curve with the plane $z = 0$, to point B, the point of intersection with the plane $z = 1$ (Fig. 30).

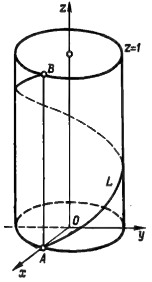

Fig. 30

Solution. Here, the line integral is of the form

$$\int_L (\mathbf{a}, d\mathbf{r}) = \int_L z\,dx + x\,dy + y\,dz.$$

The helical curve is located on a circular cylinder $x^2 + y^2 = R^2$. At point A we have $t_0 = 0$, at point B we have $t_1 = 2\pi$. Since

$$dx = -R\sin t\,dt, \quad dy = R\cos t\,dt, \quad dz = \frac{dt}{2\pi},$$

it follows that the integral is equal to

$$\int_L (\mathbf{a}, d\mathbf{r}) = \int_0^{2\pi} \left(-\frac{t}{2\pi} R\sin t + R^2\cos^2 t + \frac{R}{2\pi}\sin t\right) dt$$

$$= R^2 \int_0^{2\pi} \cos^2 t\,dt - \frac{R}{2\pi}\int_0^{2\pi} t\sin t\,dt = \pi R^2 + R$$

because

$$\int_0^{2\pi} \cos^2 t\,dt = \pi;$$

$$\int_0^{2\pi} t \sin t\,dt = -2\pi.$$

Example 3. Find the line integral of the vector (see example 2)

$$\mathbf{a} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$$

over the straight line AB (see Fig. 30) in the direction from point A to point B.

Solution. Since the straight line AB (the generatrix of the cylinder $x^2 + y^2 = R^2$) is located in the xz-plane and passes through the point A $(R, 0, 0)$, it follows that $y = 0$, $x = R$, $dx = 0$, and for the radius vector $\mathbf{r}$ of the points of AB we will have $\mathbf{r} = R\mathbf{i} + z\mathbf{k}$, $d\mathbf{r} = dz \cdot \mathbf{k}$. Therefore the scalar product

$$(\mathbf{a}, d\mathbf{r}) = z\,dx + x\,dy + y\,dz$$

on AB will be zero.

Hence, the desired line integral

$$\int_L (\mathbf{a}, d\mathbf{r}) = \int_{AB} (\mathbf{a}, d\mathbf{r})$$

on AB will be zero.

From examples 2 and 3 it follows that in the general case a line integral in a vector field depends not only on the initial and terminal points of the path of integration but also on the shape of the path.

Example 4. Compute the work done by the force field

$$\mathbf{F} = y\mathbf{i} + x\mathbf{j} + (x + y + z)\,\mathbf{k}$$

along the segment AB of a straight line passing through the points M_1 $(2, 3, 4)$ and $M_2(3, 4, 5)$.

Solution. The work done by the force field will be equal to the line integral along M_1M_2:

$$A = \int_{M_1M_2} (\mathbf{F}, d\mathbf{r}) = \int_{M_1M_2} y\,dx + x\,dy + (x + y + z)\,dz,$$

Let us find the canonical equations of the straight line M_1M_2. We have

$$\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{1},$$

whence

$$\left.\begin{aligned} y&=x+1,\\ z&=x+2,\end{aligned}\right\}$$

$$dy = dx, \quad dz = dx.$$

Here, x varies from 2 to 3 (since the abscissa of M_1 is equal to 2 and the abscissa of M_2 is 3). The desired work is

$$A=\int_2^3 (x+1+x+x+x+1+x+2)\,dx$$

$$=\int_2^3 (5x+4)\,dx=\frac{33}{2}.$$

166. In the plane vector field

$$\mathbf{a}=\frac{y^2\mathbf{i}-x^2\mathbf{j}}{\sqrt{x^2+y^2}}$$

compute the line integral along the semicircle

$$x = R\cos t, \quad y = R\sin t \quad (0 \leqslant t \leqslant \pi).$$

167. In the plane vector field

$$\mathbf{a} = (x^2 + y^2)\,\mathbf{i} + (x^2 - y)\,\mathbf{j}$$

compute the line integral over the curve $y = |x|$ from the point $(-1, 1)$ to the point $(2, 2)$.

168. In the plane vector field

$$\mathbf{a} = (x^2 - 2xy)\,\mathbf{i} + (y^2 - 2xy)\,\mathbf{j}$$

compute the line integral:

(a) along the parabola $y = x^2$ from the point $(-1, 1)$ to the point $(1, 1)$;

(b) along a segment of the straight line joining the points $(-1, 1)$ and $(1, 1)$.

169. Compute the work of the force field $\mathbf{F} = 2xy\mathbf{i} + x^2\mathbf{j}$ along an arc of the circle $x^2 + y^2 = 1$ from the point $(1, 0)$ to the point $(0, 1)$ counterclockwise.

170. Compute the work of the force field

$$\mathbf{F} = (x^2 + 2xy)\,\mathbf{i} + (x^2 + y^2)\,\mathbf{j}$$

along the parabola $y = x^2$ from the point (0, 0) to the point (1, 1).

171. In the vector field

$$\mathbf{a} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2 - x - y + 2z}}$$

compute the line integral along a segment of the straight line from the point (1, 1, 1) to the point (4, 4, 4).

172. In the vector field

$$\mathbf{a} = (y^2 - z^2)\,\mathbf{i} + 2yz\mathbf{j} - x^2\mathbf{k}$$

compute the line integral over the line

$$\left.\begin{array}{l} x = t \\ y = t^2, \\ z = t^3 \end{array}\right\} \quad (0 \leqslant t \leqslant 1)$$

in the direction of increasing values of the parameter t.

173. In the vector field

$$\mathbf{a} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$

compute the line integral along a turn of the helical curve

$$\left.\begin{array}{l} x = a\cos t, \\ y = a\sin t, \\ z = bt \end{array}\right\} \quad (0 \leqslant t < 2\pi)$$

in the direction of increasing values of the parameter t.

174. In the vector field

$$\mathbf{a} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$$

compute the line integral in the direction from the point (0, 0, 0) to the point (1, 1, 1) along the line segment between the two points.

Computing the circulation of a vector field

Definition 2. The *circulation* C of a vector field $\mathbf{a} = \mathbf{a}(M)$ is a line integral taken around a closed oriented curve L Thus, by definition,

$$C = \oint_L (\mathbf{a}, d\mathbf{r}),$$

where the symbol $\oint\limits_L$ denotes the integral around the closed curve L.

If the vector field $\mathbf{a} = \mathbf{a}(M)$ is given in coordinate form,

$$\mathbf{a} = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k},$$

then the circulation of the vector field is

$$C = \oint\limits_L P\,dx + Q\,dy + R\,dz.$$

For the positive direction of traverse around the closed curve L we take the direction in which the region bounded by the curve is on the left.

Example 5. Compute the circulation of the vector field

$$\mathbf{a} = -y^3\mathbf{i} + x^3\mathbf{j}$$

around the ellipse L: $x^2/a^2 + y^2/b^2 = 1$.

Solution. By the definition of circulation we have

$$C = \oint\limits_L (\mathbf{a}, d\mathbf{r}) = \oint\limits_L -y^3\,dx + x^3\,dy. \tag{3}$$

The parametric equations of this ellipse are of the form

$$\left.\begin{aligned} x &= a\cos t, \\ y &= b\sin t \end{aligned}\right\} \quad (0 \leqslant t < 2\pi), \tag{4}$$

whence

$$dx = -a\sin t\,dt, \quad dy = b\cos t\,dt. \tag{5}$$

Substituting (4) and (5) into (3), we get

$$C = ab\int\limits_0^{2\pi} (b^2\sin^4 t + a^2\cos^4 t)\,dt = \frac{3}{4}\pi ab\,(a^2 + b^2)$$

since

$$\int\limits_0^{2\pi} \sin^4 t\,dt = \frac{1}{4}\int\limits_0^{2\pi} (1 - \cos 2t)^2\,dt$$

$$= \frac{1}{4}\int\limits_0^{2\pi} \left(1 - 2\cos 2t + \frac{1 + \cos 4t}{2}\right) dt$$

$$=\frac{1}{4}\int_0^{2\pi}\left(\frac{3}{2}-2\cos 2t+\frac{1}{2}\cos 4t\right)dt=\frac{1}{4}\int_0^{2\pi}\frac{3}{2}\,dt=\frac{3}{4}\,\pi.$$

Similarly, we find that

$$\int_0^{2\pi}\cos^4 t\,dt=\frac{3}{4}\,\pi.$$

Example 6. Calculate the circulation of the vector field

$$\mathbf{a}=ye^{xy}\mathbf{i}+xe^{xy}\mathbf{j}+xyz\mathbf{k}$$

around the curve L obtained by cutting the cone $x^2+y^2=(z-1)^2$ with the coordinate planes (Fig. 31).

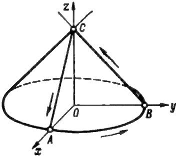

Fig. 31.

Solution. The curve L consists of two parts BC and CA located in the coordinate yz- and xz-planes respectively, and the arc $\widehat{AB}$ of the circle $\left.\begin{array}{l} x^2+y^2=1 \\ z=0 \end{array}\right\}$. Therefore the circulation of the given vector field will be

$$C=\oint_L(\mathbf{a},d\mathbf{r})=\int_{BC}(\mathbf{a},d\mathbf{r})+\int_{CA}(\mathbf{a},d\mathbf{r})+\int_{\widehat{AB}}(\mathbf{a},d\mathbf{r}).$$

(1) On the line segment BC we have

$$x=0,\ dx=0;\ \ z=1-y,\ dz=-dy;\ \ 1\geqslant y\geqslant 0.$$

Consequently,

$$\int_{BC} (\mathbf{a}, d\mathbf{r}) = \int_{BC} y\,dx = 0.$$

(2) On the line segment CA we have

$$y = 0,\ dy = 0;\quad z = 1 - x,\ dz = -dx;\quad 0 \leqslant x \leqslant 1$$

and so

$$\int_{CA} (\mathbf{a}, d\mathbf{r}) = \int_{CA} x\,dy = 0.$$

(3) On the arc $\widehat{AB}$ of the circle $\left.\begin{matrix} x^2+y^2=1, \\ z=0 \end{matrix}\right\}$ we have $z=0$, $dz=0$, which means that

$$\int_{\widehat{AB}} (\mathbf{a}, d\mathbf{r}) = \int_{\widehat{AB}} e^{xy}(y\,dx + x\,dy) = \int_{\widehat{AB}} e^{xy} d(xy)$$

$$= \int_{\widehat{AB}} d(e^{xy}) = e^{xy}\Big|_{A(1,\,0)}^{B(0,\,1)} = 1 - 1 = 0.$$

The desired circulation of the vector field is zero.

Example 7. Compute the circulation of the vector field

$$\mathbf{a} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$$

if

$$L: \begin{cases} x^2+y^2=1, \\ x+y+z=1. \end{cases}$$

Solution. We have

$$C = \oint_L (\mathbf{a}, d\mathbf{r}) = \int_L xy\,dx + yz\,dy + xz\,dz.$$

The curve L is an ellipse obtained by cutting the cylinder $x^2 + y^2 = 1$ by the plane $x + y + z = 1$. Let us find the parametric equations of the curve. The projection of any point of the curve on the xy-plane lies on the circle $x^2 + y^2 = 1$. From this we obtain $x = \cos t$, $y = \sin t$. But the ellipse lies in the plane $x + y + z = 1$, whence $z = 1 - x - y$ or $z = 1 - \cos t - \sin t$. Thus

the parametric equations of the curve L are:

$$\left.\begin{array}{l} x=\cos t, \\ y=\sin t, \\ z=1-\cos t-\sin t \end{array}\right\} \quad (0\leqslant t<2\pi).$$

From this we find

$$dx = -\sin t\,dt,\ dy = \cos t\,dt,\ dz = (\sin t - \cos t)dt$$

and so the circulation is

$$C=\int_0^{2\pi}[-\cos t\sin^2 t+\sin t\,(1-\cos t-\sin t)\cos t$$
$$+\cos t\,(1-\cos t-\sin t)(\sin t-\cos t)]\,dt$$
$$=\int_0^{2\pi}(-3\sin^2 t\cos t+\sin 2t-\cos^2 t\sin t-\cos^2 t+\cos^3 t)\,dt$$
$$=-\int_0^{2\pi}\cos^2 t\,dt=-\pi.$$

Compute the circulation C of a vector $\mathbf{a}$ around the given curve L:

175. $\mathbf{a} = (xz + y)\,\mathbf{i} + (yz - x)\,\mathbf{j} - (x^2 + y^2)\,\mathbf{k}$;

$$L: \begin{cases} x^2+y^2=1, \\ z=3. \end{cases}$$

176. $\mathbf{a} = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$;

$$L: \begin{cases} x^2+y^2+z^2=R^2, \\ x^2+y^2=Rx \end{cases} \quad (z\geqslant 0).$$

177. $\mathbf{a} = (2x + z)\,\mathbf{i} + (2y - z)\,\mathbf{j} + xyz\mathbf{k}$. L is the line of intersection of the paraboloid of revolution $x^2 + y^2 = 1 - z$ with the coordinate planes.

178. Show that if in a vector field the circulation of a vector around any closed circuit is equal to zero, then there can be no closed vector lines in such a field.

Sec. 15. The curl (rotation) of a vector field

Suppose we have the vector field

$$\mathbf{a}(M) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}.$$

We will assume that the coordinates P, Q, R of the vector $\mathbf{a}(M)$ are continuous and have continuous partial derivatives of the first order with respect to all its arguments.

Definition 1. The *curl* (or *rotation*) of a vector $\mathbf{a}(M)$ is a vector (symbolized: curl $\mathbf{a}(M)$ or rot $\mathbf{a}(M)$) defined by the equation

$$\operatorname{curl} \mathbf{a} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \quad (1)$$

or, in easy-to-remember symbolic form,

$$\operatorname{curl} \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}. \quad (2)$$

This determinant is ordinarily expanded in terms of elements of the first row, in which case the operations of multiplication of elements of the second row by elements of the third row are regarded as operations of differentiation; for example,

$$\frac{\partial}{\partial x} \cdot Q = \frac{\partial Q}{\partial x}.$$

Definition 2. If in some region G we have curl $\mathbf{a} = 0$, then the field of the vector $\mathbf{a}$ in G is said to be *irrotational*.

Example 1. Find the curl of the vector

$$\mathbf{a} = (x + z)\,\mathbf{i} + (y + z)\,\mathbf{j} + (x^2 + z)\,\mathbf{k}.$$

Solution. Using formula (2), we obtain

$$\operatorname{curl} \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+z & y+z & x^2+z \end{vmatrix}.$$

Expanding the determinant in terms of elements of the first row and regarding the operation of multiplication, say $\partial/\partial y$ by $x^2 + z$, as an operation of partial differentiation, we obtain

$$\operatorname{curl} \mathbf{a} = -\mathbf{i} - (2x - 1)\,\mathbf{j}.$$

Example 2. Find the curl of **H**, the intensity vector of a magnetic field under the conditions of example 3 of Sec. 10.

Solution. The magnetic-field intensity vector **H** is

$$\mathbf{H} = \frac{2}{\rho^2}[\mathbf{I}, \mathbf{r}]$$

or

$$\mathbf{H} = \frac{2}{\rho^2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & I \\ x & y & z \end{vmatrix} = -\frac{2I}{\rho^2}y\mathbf{i} + \frac{2I}{\rho^2}x\mathbf{j},$$

where $\rho^2 = x^2 + y^2$, whence, by (2),

$$\operatorname{curl} \mathbf{H} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{2Iy}{x^2+y^2} & \frac{2Ix}{x^2+y^2} & 0 \end{vmatrix} = \left[\frac{\partial}{\partial x}\left(\frac{2Ix}{x^2+y^2}\right)\right.$$

$$\left. + \frac{\partial}{\partial y}\left(\frac{2Iy}{x^2+y^2}\right)\right]\mathbf{k} = 2I\left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right]\mathbf{k} = 0$$

$$(x^2 + y^2 \neq 0).$$

Thus, curl $\mathbf{H} = 0$ everywhere except the z-axis, at the points of which the last formulas are meaningless (the denominator vanishes), that is, the field of the vector **H** is irrotational everywhere outside the points of the z-axis.

Find the curl of the following vectors:

179. $\mathbf{a} = (x^2 + y^2)\,\mathbf{i} + (y^2 + z^2)\,\mathbf{j} + (z^2 + x^2)\,\mathbf{k}$.

180. $\mathbf{a} = z^3\mathbf{i} + y^3\mathbf{j} + x^3\mathbf{k}$.

181. $\mathbf{a} = \frac{1}{2}(-y^2\mathbf{i} + x^2\mathbf{j})$.

182. Show that if the coordinates of the vector $\mathbf{a}\,(M)$ have continuous partial derivatives of the second order, then

$$\operatorname{div} \operatorname{curl} \mathbf{a} = 0,$$

that is, the field of the vector curl $\mathbf{a}(M)$ is a solenoidal field.

183. Show that

(a) $\operatorname{curl}(\mathbf{a} \pm \mathbf{b}) = \operatorname{curl}\mathbf{a} \pm \operatorname{curl}\mathbf{b}$,

(b) $\operatorname{curl}(\lambda\mathbf{a}) = \lambda \operatorname{curl}\mathbf{a}$.

where λ is a numerical constant.

184. Show that if $u = u(M)$ is a scalar function and $\mathbf{a} = \mathbf{a}(M)$ is a vector, then

$$\operatorname{curl}(u\mathbf{a}) = u \operatorname{curl}\mathbf{a} + [\operatorname{grad} u, \mathbf{a}].$$

185. Show that if $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, and $\mathbf{r}$ is the radius vector of the point $M(x, y, z)$, then

$$\operatorname{curl}(\mathbf{r}, \mathbf{a})\,\mathbf{b} = [\mathbf{a}, \mathbf{b}].$$

186. Show that

$$\operatorname{curl}(r\mathbf{a}) = \frac{1}{r}[\mathbf{r}, \mathbf{a}],$$

where $\mathbf{a}$ is a constant vector and $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

187. Show that $\operatorname{curl}(f(r)\,\mathbf{a}) = \frac{f'(r)}{r}[\mathbf{r}, \mathbf{a}]$, where $f(r)$ is an arbitrary differentiable function of its argument and $\mathbf{a}$ is a constant vector.

188. Show that the vector field $\mathbf{a} = f(r)\,\mathbf{r}$ is irrotational that is, $\operatorname{curl}\mathbf{a} \equiv 0$.

189. Show that

$$\operatorname{div}[\mathbf{a}, \mathbf{b}] = (\mathbf{b}, \operatorname{curl}\mathbf{a}) - (\mathbf{a}, \operatorname{curl}\mathbf{b}).$$

190. Show that the curl of a field of linear velocities $\mathbf{v}$ of a rotating solid is a constant vector in the direction parallel to the axis of rotation, the modulus of which is equal to twice the angular velocity of rotation: $\operatorname{curl}\mathbf{v} = 2\boldsymbol{\omega}$.

191. Determine the angular velocity $\boldsymbol{\omega}$ of rotation of a solid about a fixed axis passing through some point of the solid if its linear velocity is

$$\mathbf{v} = 2x\mathbf{i} + y^2\mathbf{j} + xz\mathbf{k}.$$

192. Show that the field of the curl of the vector $\mathbf{a}(M)$ is free of sources and sinks.

193. What must the function $f(x, z)$ be so that the curl

of the vector field

$$\mathbf{a} = yz\mathbf{i} + f(x, z)\,\mathbf{j} + xy\mathbf{k}$$

is coincident with the vector $\mathbf{k} - \mathbf{i}$?

Sec. 16. Stokes' theorem

Suppose the coordinates of a vector

$$\mathbf{a}(M) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

are continuous and have continuous partial derivatives.

Theorem. *The circulation of the vector* $\mathbf{a}$ *around a closed contour* L *is equal to the flux of the curl of the vector through any surface* Σ *spanning the contour* L:

$$\oint_L (\mathbf{a}, d\mathbf{r}) = \iint_\Sigma (\operatorname{curl}\,\mathbf{a}, \mathbf{n}^0)\, d\sigma. \tag{1}$$

It is assumed that the orientation of the normal $\mathbf{n}^0$ to the surface Σ is matched with the orientation of the contour L so that, when viewed from the end of the normal, the contour is traversed in the chosen direction counterclockwise.

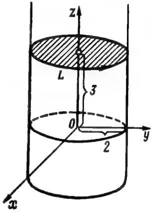

Fig. 32

Example 1. Compute the circulation of the vector

$$\mathbf{a} = y\mathbf{i} + x^2\mathbf{j} - z\mathbf{k}$$

around the contour L: $\begin{cases} x^2 + y^2 = 4, \\ z = 3 \end{cases}$ (1) directly and (2) via the Stokes theorem.

Solution. (1) The contour L is a circle of radius $R = 2$ lying in the plane $z = 3$ (Fig. 32). We choose the orientation as shown in the drawing. The parametric equations of

the curve L are

$$\left.\begin{aligned} x &= 2\cos t, \\ y &= 2\sin t, \\ z &= 3 \end{aligned}\right\} \quad (0 \leqslant t < 2\pi),$$

so that

$$dx = -2\sin t\,dt, \quad dy = 2\cos t\,dt, \quad dz = 0.$$

For the circulation of the vector **a** we have

$$C = \int_0^{2\pi} [2\sin t(-2\sin t) + 4\cos^2 t\, 2\cos t - 3\cdot 0]\,dt = -4\pi.$$

(2) To compute the circulation via the Stokes theorem, choose some surface Σ spanning the contour L. For Σ it is natural to take a circle having L as its boundary. According to the chosen orientation of the contour the normal $\mathbf{n}^0$ to the circle has to be taken equal to $\mathbf{k}$: $\mathbf{n}^0 = \mathbf{k}$. Then

$$\operatorname{curl} \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x^2 & -z \end{vmatrix} = (2x-1)\,\mathbf{k}.$$

Therefore, by the Stokes theorem,

$$C = \iint_\Sigma (\operatorname{curl} \mathbf{a}, \mathbf{n}^0)\,d\sigma = \iint_\Sigma (2x-1)\,d\sigma$$

$$= \int_0^{2\pi} d\varphi \int_0^2 (2\rho\cos\varphi - 1)\,\rho\,d\rho = -2\pi \frac{\rho^2}{2}\Big|_0^2 = -4\pi.$$

194. Show that the flux of the rotor through an open surface spanning a given contour does not depend on the shape of the surface.

Find the circulations of the vectors around the indicated contours (1) directly and (2) via the Stokes theorem.

195. $\mathbf{a} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$; $\quad L: \begin{cases} x^2 + y^2 = 4, \\ z = 0. \end{cases}$

196. $\mathbf{a} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$; $\quad L: \begin{cases} x^2 + y^2 + z^2 = 4, \\ x^2 + y^2 = z^2 \quad (z \geqslant 0). \end{cases}$

197. $\mathbf{a} = 2xz\mathbf{i} - y\mathbf{j} + z\mathbf{k}$ around a contour formed by intersection of the plane $x + y + 2z = 2$ with the coordinate planes.

198. $\mathbf{a} = y\mathbf{i} - x\mathbf{j} + (x + y)\mathbf{k}$; L: $\begin{cases} z = x^2 + y^2, \\ z = 1. \end{cases}$

199. $\mathbf{a} = z^2\mathbf{i}$; L: $\begin{cases} x^2 + y^2 + z^2 = 16, \\ x = 0, y = 0, z = 0. \end{cases}$

200. $\mathbf{a} = zy^2\mathbf{i} + xz^2\mathbf{j} + x^2y\mathbf{k}$; L: $\begin{cases} x = y^2 + z^2, \\ x = 9. \end{cases}$

201. $\mathbf{a} = y^2\mathbf{i} + z^2\mathbf{j}$; L: $\begin{cases} x^2 + y^2 = 9, \\ 3y + 4z = 5. \end{cases}$

202. $\mathbf{a} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$; L: $\begin{cases} x^2 + y^2 + z^2 = 1, \\ x = z. \end{cases}$

203. Given a vector field of velocities $\mathbf{v}$ of the points of a solid rotating with a constant angular velocity ω about the z-axis. Calculate the circulation of the field around the circle

$$L: \begin{cases} x = a\cos t, \\ y = a\sin t, \quad (0 \leqslant t < 2\pi), \\ z = 0 \end{cases}$$

directly and via the Stokes theorem.

From the Stokes theorem we find that the projection of the vector curl $\mathbf{a}$ on any direction $\mathbf{n}$ is independent of the choice of the system of coordinates and is equal to the surface density of the circulation of the vector $\mathbf{a}$ around the contour of the area perpendicular to that direction:

$$\operatorname{pr}_{\mathbf{n}} \operatorname{curl} \mathbf{a}|_M = (\operatorname{curl} \mathbf{a}, \mathbf{n}^0)\Big|_M = \lim_{(\Sigma)\to M} \frac{\oint\limits_L (\mathbf{a}, d\mathbf{r})}{S}. \qquad (2)$$

Here, (Σ) is a plane area perpendicular to the vector $\mathbf{n}$; S is the area of (Σ); L is the contour of the area and is oriented so that the traverse of the contour is counterclockwise as seen from the end of the vector $\mathbf{n}$; the nota-

tion $(\Sigma) \to M$ means that the area (Σ) contracts to the point M at which we consider the vector curl **a**, and the direction of the normal **n** to that area is always the same.

Example 2. Compute the density of the circulation of the vector $\mathbf{a} = y\mathbf{i}$ around the circle

$$L: \begin{cases} x = a\cos t, \\ y = a\sin t, \quad (0 \leqslant t < 2\pi), \\ z = 0 \end{cases}$$

at the centre of the circle, $M(0, 0, 0)$, in the positive direction of the z-axis.

Solution. Here, (Σ) is a circle of radius a with centre at M, so that $S = \pi a^2$.

The desired density of the circulation is

$$\mu_M = \lim_{a\to 0} \frac{1}{\pi a^2} \oint_L (\mathbf{a}, d\mathbf{r}) = \lim_{a\to 0} \frac{1}{\pi a^2} \oint_L y\,dx$$

$$= \lim_{a\to 0} \frac{1}{\pi a^2} \int_0^{2\pi} (-a^2)\sin^2 t\,dt = -1.$$

On the other hand,

$$\operatorname{curl} \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} = -\mathbf{k}$$

and, hence,

$$(\operatorname{curl} \mathbf{a}, \mathbf{n}^0)\,|_M = (-\mathbf{k}, \mathbf{k}) = -1,$$

which, by virtue of (2), confirms the correctness of the result.

204. Compute the density of the circulation of the vector $\mathbf{a} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ around the circle L: $\{y = a\cos t,\ z = a\sin t,\ x = 0,\ (0 \leqslant t < 2\pi)\}$ at the centre of the circle, $C(0, 0, 0)$, in the positive direction of the x-axis.

205. Compute the density of the circulation of the vector $\mathbf{a} = 2y\mathbf{i} + 5xz\mathbf{j}$ around the ellipse L: $\{x = a\cos t,\ y = b\sin t,\ z = 1\ (0 \leqslant t < 2\pi)\}$ at the centre $C(0, 0, 1)$ of the ellipse in the positive direction of the z-axis.

Sec. 17. The independence of a line integral of the path of integration. Green's formula

Definition. A region G of three-dimensional space is said to be *simply connected* (more precisely, it is a simply connected plane region) if any closed contour lying in the region can be spanned by a surface lying entirely in G. For example, the whole of three-dimensional space and the interior of a sphere are simply connected regions; the interior of a torus and three-dimensional space with a straight line deleted are not simply connected regions.

Theorem. *In order that the line integral*

$$\int_L (\mathbf{a}, d\mathbf{r}) = \int_L P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$$

should not depend on the form of the path of integration L, it is necessary and sufficient that the vector field

$$\mathbf{a} = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

be irrotational, that is,

$$\operatorname{curl} \mathbf{a}(M) = 0.$$

It is assumed here that the coordinates $P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$ of the vector $\mathbf{a}$ have continuous partial derivatives of the first order and the domain of definition of the vector $\mathbf{a}(M)$ is simply connected.

In that case, the line integral $\int\limits_L (\mathbf{a}, d\mathbf{r})$ will depend solely on the position of the initial and terminal points of the path of integration L.

If the theorem is complied with, the *circulation of the vector* $\mathbf{a}(M)$ *around any closed contour C located in the field of the vector* $\mathbf{a}(M)$ *is equal to zero*:

$$\oint_C (\mathbf{a}, d\mathbf{r}) = 0.$$

Example 1. Show that in the field of the vector

$$\mathbf{a} = xy^2z\mathbf{i} + x^2yz\mathbf{j} + \frac{1}{2}x^2y^2\mathbf{k}$$

the line integral $\int_L (\mathbf{a}, d\mathbf{r})$ is independent of the shape of the path of integration L.

Solution. The coordinates of the vector $\mathbf{a}$ are everywhere continuous functions so that the domain of definition G of the vector $\mathbf{a}$ is the entire space (a simply connected region). In this region we have

$$\operatorname{curl} \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z & x^2yz & \frac{1}{2}x^2y^2 \end{vmatrix} \equiv 0.$$

Consequently, the line integral

$$\int_L (\mathbf{a}, d\mathbf{r}) = \int_L xy^2z\,dx + x^2yz\,dy + \frac{1}{2}x^2y^2\,dz$$

is independent of the shape of the path of integration L.

In particular, for the plane vector field

$$\mathbf{a}(M) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} \tag{1}$$

we have

$$\operatorname{curl} \mathbf{a}(M) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

Therefore, for the plane vector field (1) that is defined in a simply connected region G, the condition $\operatorname{curl} \mathbf{a}(M) = 0$ is written in coordinate form thus: $\partial P/\partial y = \partial Q/\partial x$.

This means that *in order for a line integral*

$$\int_L P(x, y)\,dx + Q(x, y)\,dy$$

in a plane field defined in a simply connected region G to be independent of the shape of the path of integration, it is necessary and sufficient that the following relation hold:

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}.$$

Remark. The requirement that the region G, where the vector $\mathbf{a} = \mathbf{a}(M)$ is defined, be simply connected is essen-

tial. If the region G is nonsimply connected, then, provided that curl $\mathbf{a}(M) \equiv 0$, the line integral may depend on the shape of the path of integration.

Example 2. Let us consider the line integral

$$\int\limits_L \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2}.$$

Solution. The integrand is meaningless at the point $O(0,\ 0)$; and so we eliminate that point. In the remaining part of the plane (which is then a nonsimply connected region), the coefficients of dx and dy are continuous and have continuous partial derivatives, and the following identity holds true:

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) \equiv \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right).$$

On the other hand, if we compute the integral around the circle L: $x^2 + y^2 = R^2$, then by parametrizing the equation of the circle we get

$$\oint\limits_L \frac{-y\,dx+x\,dy}{x^2+y^2} = \int\limits_0^{2\pi} \frac{R^2\sin^2 t+R^2\cos^2 t}{R^2}\,dt = \int\limits_0^{2\pi} dt = 2\pi.$$

We find that the circulation is nonzero and, hence, the line integral depends on the path of integration.

Determine in which of the vector fields indicated below the integral is independent of the shape of the path of integration:

206. $\mathbf{a} = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$.

207. $\mathbf{a} = \dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{1+x^2+y^2+z^2}}$.

208. $\mathbf{a} = \dfrac{y\mathbf{i}-x\mathbf{j}+z\mathbf{k}}{x^2+y^2+z^2}$.

Green's formula

Given in a region D with boundary L a plane vector field

$$\mathbf{a} = P(x,\ y)\,\mathbf{i} + Q(x,\ y)\,\mathbf{j},$$

where the coordinates $P(x,\ y)$, $Q(x,\ y)$ are continuous and have continuous partial derivatives $\partial P/\partial y$, $\partial Q/\partial x$.

Then Green's formula,

$$\oint_L P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy, \tag{2}$$

holds true. Here, the boundary L is traversed so that the region D is on the left.

Fig. 33

The region D may also be nonsimply connected so that the boundary may consist of several components (Fig. 33). In that case, the integral

$$\int_L P\,dx + Q\,dy$$

is understood to mean the sum of the integrals over all components of the boundary of D.

Green's formula (2) is a special case of the Stokes theorem (see Sec. 16).

In certain cases, Green's formula permits simplifying the computation of the circulation of a vector field.

Example 3. Compute the circulation of the vector

$$\mathbf{a} = \sqrt{1+x^2+y^2}\,\mathbf{i} + y\left[xy + \ln\left(x + \sqrt{1+x^2+y^2}\right)\right]\mathbf{j}$$

around the circle $x^2 + y^2 = R^2$.

Solution. The circulation of the given vector is

$$C = \oint_L (a, d\mathbf{r}) = \oint_L \sqrt{1+x^2+y^2}\,dx$$

$$+ y\left[xy + \ln\left(x + \sqrt{1+x^2+y^2}\right)\right] dy.$$

Here

$$P=\sqrt{1+x^2+y^2}, \quad Q=xy^2+y\ln\left(x+\sqrt{1+x^2+y^2}\right).$$

We find the partial derivatives

$$\frac{\partial P}{\partial y}=\frac{y}{\sqrt{1+x^2+y^2}}, \quad \frac{\partial Q}{\partial x}=y^2+\frac{y}{\sqrt{1+x^2+y^2}}.$$

Using Green's formula, we obtain

$$C=\iint\limits_D\left(y^2+\frac{y}{\sqrt{1+x^2+y^2}}-\frac{y}{\sqrt{1+x^2+y^2}}\right)dx\,dy$$

$$=\iint\limits_D y^2\,dx\,dy.$$

Passing to polar coordinates,

$$x=\rho\cos\varphi, \quad y=\rho\sin\varphi,$$

we have

$$C=\iint\limits_D \rho^2\sin^2\varphi\rho\,d\rho\,d\varphi=\iint\limits_D \rho^3\sin^2\varphi\,d\rho\,d\varphi.$$

Since $0\leqslant\varphi<2\pi$, $0\leqslant\rho\leqslant R$, it follows that

$$C=\int\limits_0^{2\pi}\sin^2\varphi\,d\varphi\int\limits_0^R\rho^3\,d\rho=\frac{\pi R^4}{4}.$$

Compute the circulation of the following vectors around the given contours using Green's formula:

209. $\mathbf{a}=(y+x)\,\mathbf{i}+(y-x)\,\mathbf{j}$; L: $x+y=1$, $x=0$, $y=0$.

210. $\mathbf{a}=(x-y^2)\,\mathbf{i}+2xy\,\mathbf{j}$; L: $y=x$, $y=x^2$.

211. $\mathbf{a}=x\ln(1+y^2)\,\mathbf{i}+\frac{x^2y}{1+y^2}\,\mathbf{j}$; L: $x^2+y^2=2x$.

212. $\mathbf{a}=y^2\mathbf{i}-x^2\mathbf{j}$; L: $x+y=-1$, $x=0$, $y=0$.

213. $\mathbf{a}=\dfrac{\left(3x-y^3\sqrt{1+x^2+4y^3}\right)\mathbf{i}+\left(18y^2+x^3\sqrt{1+x^2+4y^3}\right)\mathbf{j}}{3\sqrt{1+x^2+4y^3}}$;

$$L:\ x^2+y^2=1.$$

214. Use Green's formula to compute the difference between the integrals

$$I_1=\int\limits_{AmB}(x+y)^2\,dx-(x-y)^2\,dy$$

and

$$I_2 = \int\limits_{AnB} (x+y)^2\,dx - (x-y)^2\,dy,$$

where AmB is a line segment joining the points A (0, 0) and B (1, 1), and AnB is an arc of the parabola $y = x^2$.

215. Prove that the integral

$$\oint\limits_L (2x+y)\,dx + 2x\,dy,$$

where L is a closed contour, yields the area of the region bounded by that contour.

216. Using Green's formula, compute the line integral $\int\limits_L (\mathbf{a}, d\mathbf{r})$ in the vector field

$$\mathbf{a} = (e^x \sin y - y)\,\mathbf{i} + (e^x \cos y - 1)\,\mathbf{j},$$

where the curve L is the upper semicircle $x^2 + y^2 = 2x$ traversed in the direction from the point A (2, 0) to the point O (0, 0).

CHAPTER IV

POTENTIAL FIELDS

Sec. 18. The criterion for the potentiality of a vector field

Definition. A vector field

$$\mathbf{a}(M) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

specified in a spatial region V is said to be *potential* if there exists a scalar function $\varphi(M)$ such that at all points of V the following equality holds:

$$\mathbf{a}(M) = \operatorname{grad} \varphi(M). \tag{1}$$

The function $\varphi(M) = \varphi(x, y, z)$ that satisfies (1) in V is termed the *potential* (or the *potential function*) of the vector field $\mathbf{a}$.

The relation (1) is equivalent to the following three scalar equations:

$$P(x, y, z) = \frac{\partial \varphi}{\partial x}, \quad Q(x, y, z) = \frac{\partial \varphi}{\partial y}, \quad R(x, y, z) = \frac{\partial \varphi}{\partial z}. \tag{2}$$

The potential of a field is not defined uniquely but only up to an additive constant.

Remark. For force fields, the function $\varphi(M)$ is ordinarily called a *force function*, and the potential is the function $-\varphi(M)$.

Example 1. (The electrostatic field of a point charge.) Show that the field of electric intensity $\mathbf{E}$ set up by a point charge q located at the coordinate origin,

$$\mathbf{E} = \frac{q}{r^3}\,\mathbf{r}, \quad r = \sqrt{x^2 + y^2 + z^2},$$

is a potential field.

Solution. The problem is posed thus: show that there exists a function $\varphi(x, y, z)$ such that relations (2) hold. In our case, we have

$$P(x, y, z)=\frac{qx}{r^3}, \quad Q(x, y, z)=\frac{qy}{r^3}, \quad R(x, y, z)=\frac{qz}{r^3}.$$

Since

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right)=-\frac{1}{r^2}\frac{\partial r}{\partial x}=-\frac{x}{r^3}$$

and, analogously,

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right)=-\frac{y}{r^3}, \quad \frac{\partial}{\partial z}\left(\frac{1}{r}\right)=-\frac{z}{r^3},$$

it follows that the function

$$\varphi(x, y, z)=-\frac{q}{r}=-\frac{q}{\sqrt{x^2+y^2+z^2}}$$

is the potential of the given field:

$$\operatorname{grad}\left(-\frac{q}{r}\right)=\mathbf{E}.$$

In this case, the coordinate origin (this is where the charge q is located) is a singular point of the field $\mathbf{E}$.

Theorem. *For a vector field* $\mathbf{a}(M)$ *specified in a simply connected region* V *to be potential, it is necessary and sufficient that the following condition hold at every point of* V:

$$\operatorname{curl} \mathbf{a}=0. \tag{3}$$

In other words, for a vector field specified in a simply connected region to be potential, it is necessary and sufficient that it be irrotational.

The potential $\varphi(x, y, z)$ of the vector field

$$\mathbf{a}=P(x, y, z)\,\mathbf{i}+Q(x, y, z)\,\mathbf{j}+R(x, y, z)\,\mathbf{k}$$

is defined by the formula

$$\varphi(x, y, z)=\int_{(x_0, y_0, z_0)}^{(x, y, z)} P\,dx+Q\,dy+R\,dz, \tag{4}$$

where (x_0, y_0, z_0) is some fixed point of the field and (x, y, z) is an arbitrary current point.

Example 2. Show that the field of the vector

$$\mathbf{a}=x^2\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}$$

is a potential field.

Solution. The coordinates $P = x^2$, $Q = y^2$, $R = z^2$ of the vector **a** are infinitely differentiable functions throughout the space so that **a** is an infinitely differentiable vector defined throughout three-dimensional space. We have

$$\text{curl } \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = \left(\frac{\partial}{\partial y} z^2 - \frac{\partial}{\partial z} y^2\right) \mathbf{i}$$

$$+ \left(\frac{\partial}{\partial z} x^2 - \frac{\partial}{\partial x} z^2\right) \mathbf{j} + \left(\frac{\partial}{\partial x} y^2 - \frac{\partial}{\partial y} x^2\right) \mathbf{k} = 0.$$

By virtue of the theorem given on p. 122 the field of the vector **a** is a potential field. It is readily seen that the function

$$\varphi(x, y, z) = \frac{x^3 + y^3 + z^3}{3} + C,$$

where C is an arbitrary constant, is the potential of the given field.

Check to see whether the following vector fields are potential fields:

217. $\mathbf{a} = xz\mathbf{i} + 2y\mathbf{j} + xy\mathbf{k}$.

218. $\mathbf{a} = (2xy + z^2)\mathbf{i} + (2yz + x^2)\mathbf{j} + (2xz + y^2)\mathbf{k}$.

219. $\mathbf{a} = \frac{1}{3}(x^3\mathbf{i} + y^3\mathbf{j} + xz^3\mathbf{k})$.

220. $\mathbf{a} = yz \cos xy \cdot \mathbf{i} + xz \cos xy \cdot \mathbf{j} + \sin xy \cdot \mathbf{k}$.

221. $\mathbf{a} = \ln(1 + z^2)\mathbf{i} + \ln(1 + x^2)\mathbf{j} + xz\mathbf{k}$.

222. $\mathbf{a} = \left(\frac{z}{x^2} + \frac{1}{y}\right)\mathbf{i} + \left(\frac{x}{y^2} + \frac{1}{z}\right)\mathbf{j} + \left(\frac{y}{z^2} + \frac{1}{x}\right)\mathbf{k}$.

223. $\mathbf{H} = \frac{2I}{r^2}(-y\mathbf{i} + x\mathbf{j})$, $r^2 = x^2 + y^2$, $r \neq 0$.

224. Prove that the field $\mathbf{a} = f(r) \cdot \mathbf{r}$, where $f(r)$ is a differentiable function, is a potential field.

225. Show that the vector lines in the potential field $\mathbf{a} = \text{grad } \varphi$ are perpendicular to the level surface of the function φ.

Sec. 19. Computing a line integral in a potential field

Theorem. *A line integral in a potential field* **a** *(M) is equal to the difference between the values of the potential* $\varphi(M)$ *of the field at the terminal and initial points of the path of integration:*

$$\int_{M_1}^{M_2} (\mathbf{a}, d\mathbf{r}) = \varphi(M_2) - \varphi(M_1). \tag{1}$$

Example 1. Compute the line integral in the field of the vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

along the line segment bounded by the points $M_1(-1, 0, 3)$ and $M_2(2, -1, 0)$.

Solution. We will show that the field of the given vector is a potential field. Indeed,

$$\text{curl } \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \equiv 0.$$

It is easy to see that the potential of this field is

$$\varphi(x, y, z) = \frac{x^2+y^2+z^2}{2} + C.$$

Using formula (1), we obtain

$$\int_{M_1}^{M_2} (\mathbf{a}, d\mathbf{r}) = \varphi(2, -1, 0) - \varphi(-1, 0, 3) = \frac{5}{2} - 5 = -\frac{5}{2}.$$

Note that it is immaterial what line joins the points M_1 and M_2; for fixed M_1 and M_2 the integral

$$\int_{M_1}^{M_2} (\mathbf{a}, d\mathbf{r}) = \int_{M_1}^{M_2} x\,dx + y\,dy + z\,dz$$

always has the same value.

Computing the potential of a field in Cartesian coordinates

The formula

$$\varphi(x, y, z) = \int_{(x_0, y_0, z_0)}^{(x, y, z)} P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz \tag{2}$$

may be used to find the potential function $\varphi(M) = \varphi(x, y, z)$ of a specified potential field

$$\mathbf{a}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}.$$

To do this, fix the initial point $M_0(x_0, y_0, z_0)$ and join it with the current point $M(x, y, z)$ with a broken line

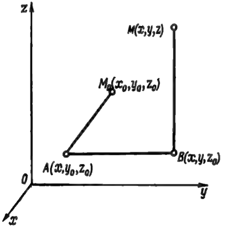

Fig. 34

M_0ABM whose segments are parallel to the coordinate axes; namely, $M_0A \parallel Ox$, $AB \parallel Oy$, $BM \parallel Oz$ (Fig. 34). Then formula (2) takes the form

$$\varphi(x, y, z) = \int_{x_0}^{x} P(x, y_0, z_0)\,dx + \int_{y_0}^{y} Q(x, y, z_0)\,dy$$

$$+ \int_{z_0}^{z} R(x, y, z)\, dz, \qquad (3)$$

where x, y, z are the coordinates of the current point on the segments of the broken line along which the integration is performed.

Example 2. Prove that the vector field

$$\mathbf{a} = (y + z)\,\mathbf{i} + (x + z)\,\mathbf{j} + (x + y)\,\mathbf{k}$$

is a potential field and find its potential.

Solution. 1st method. A necessary and sufficient condition for the potentiality of a field $\mathbf{a}(M)$ is that curl $\mathbf{a}(M)$ be zero. In our case,

$$\operatorname{curl} \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & x+z & x+y \end{vmatrix}$$
$$= (1-1)\,\mathbf{i} + (1-1)\,\mathbf{j} + (1-1)\,\mathbf{k} \equiv 0.$$

That is, the field is a potential field. The potential of the field can be found with the aid of formula (3). For the initial fixed point we take the coordinate origin $O(0, 0, 0)$. We thus have

$$\varphi(x, y, z) = \int_0^x (0+0)\, dx + \int_0^y (x+0)\, dy + \int_0^z (x+y)\, dz$$
$$= xy + xz + yz.$$

To summarize,

$$\varphi(x, y, z) = xy + xz + yz + C,$$

where C is an arbitrary constant.

2nd method. By definition, the potential $\varphi(x, y, z)$ is a scalar function for which grad $\varphi = \mathbf{a}$. This vector equality is equivalent to three scalar equalities:

$$\frac{\partial\varphi}{\partial x} = y + z, \qquad (4)$$

$$\frac{\partial\varphi}{\partial y} = x + z, \qquad (5)$$

$$\frac{\partial\varphi}{\partial z} = x + y. \qquad (6)$$

Integrating (4) with respect to x, we obtain

$$\varphi(x, y, z) = \int_0^x (y+z)\,dx = xy + xz + f(y, z), \qquad (7)$$

where $f(y, z)$ is an arbitrary differentiable function of y and z. Differentiating both sides of (7) with respect to y and taking into account (5), we obtain a relation for finding the as yet undetermined function $f(y, z)$. We have

$$\frac{\partial\varphi}{\partial y} = x + \frac{\partial f}{\partial y}$$

or

$$x + z = x + \frac{\partial f}{\partial y},$$

whence

$$z = \frac{\partial f}{\partial y}. \qquad (8)$$

Integrating (8) with respect to y, we have

$$f(y, z) = \int_0^y z\,dy = zy + F(z), \qquad (9)$$

where $F(z)$ is an as yet undetermined function of z. Substituting (9) into (7), we get

$$\varphi(x, y, z) = xy + xz + zy + F(z).$$

Differentiating this equation with respect to z and taking into account (6), we obtain an equation for finding $F(z)$:

$$x + y = x + y + \frac{dF}{dz},$$

whence $dF/dz = 0$ so that $F(z) \equiv C =$ constant.

Thus we have

$$\varphi(x, y, z) = xy + yz + zx + C.$$

3rd method. By the definition of the complete differential of the function $\varphi(x, y, z)$ we have

$$d\varphi = \frac{\partial\varphi}{\partial x}\,dx + \frac{\partial\varphi}{\partial y}\,dy + \frac{\partial\varphi}{\partial z}\,dz.$$

Substituting in place of the partial derivatives $\partial\varphi/\partial x$, $\partial\varphi/\partial y$, $\partial\varphi/\partial z$ their expressions taken from (4), (5), (6), we obtain

$$d\varphi = (y + z)\,dx + (x + z)\,dy + (x + y)\,dz$$

or, after some simple algebra,

$$d\varphi = (y\,dx + x\,dy) + (z\,dx + x\,dz) + (y\,dz + z\,dy)$$
$$= d(xy) + d(xz) + d(yz) = d(xy + xz + yz).$$

Thus

$$d\varphi = d(xy + yz + zx),$$

whence it follows that

$$\varphi(x, y, z) = xy + yz + zx + C.$$

In the following problems, establish the potentiality of the given vector fields $\mathbf{a}(M)$ and find their potentials $\varphi(M)$:

226. $\mathbf{a} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$.

227. $\mathbf{a} = (yz + 1)\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$.

228. $\mathbf{a} = (2xy + z)\mathbf{i} + (x^2 - 2y)\mathbf{j} + x\mathbf{k}$.

229. $\mathbf{a} = \dfrac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{x+y+z}$.

230. $\mathbf{a} = \dfrac{yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}}{1 + x^2y^2z^2}$.

231. $\mathbf{a} = \dfrac{\mathbf{r}}{r}$.

232. $\mathbf{a} = \dfrac{\mathbf{r}}{r^3}$.

233. $\mathbf{a} = r \cdot \mathbf{r}$.

When the region Ω is a star with centre at the coordinate origin $O(0, 0, 0)$*, the potential $\varphi(M)$ of some

* The region Ω is called a star-shaped region relative to the point O belonging to Ω if any ray emanating from that point cuts the boundary of the region at one point at most. For example, star-shaped regions in the plane are the plane itself, a parallelogram, a circle; in three-dimensional space, the space itself, a parallelepiped, a sphere.

vector field $\mathbf{a} = \mathbf{a}(M)$ at the point $M(x, y, z)$ may be found from the formula

$$\varphi(M) = \int_0^1 (\mathbf{a}(M'),\ \mathbf{r}(M))\,dt + C, \quad C = \text{constant}, \tag{10}$$

where $\mathbf{r}(M) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the radius vector of the point $M(x, y, z)$, and the point $M'(tx, ty, tz)$ for $0 \leqslant t \leqslant 1$ runs over the segment OM of the straight line passing through the points O and M.

Example 3. Find the potential of the vector field

$$\mathbf{a} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}.$$

Solution. It is readily seen that $\operatorname{curl} \mathbf{a} \equiv 0$, which means the given vector field is a potential field. This field is defined throughout three-dimensional space and is star-shaped with centre at the coordinate origin $O(0, 0, 0)$, and therefore to find the potential we take advantage of formula (10). Since in this case

$$\mathbf{a}(M') = \mathbf{a}(tx, ty, tz) = t^2yz\mathbf{i} + t^2xz\mathbf{j} + t^2xy\mathbf{k},$$

it follows that the scalar product of the vectors $\mathbf{a}(M')$ and $\mathbf{r}(M)$ is equal to

$$(\mathbf{a}(M'),\ \mathbf{r}(M)) = t^2(xyz + xyz + xyz) = 3t^2xyz.$$

The desired potential is

$$\varphi(M) = \int_0^1 (\mathbf{a}(M'),\ \mathbf{r}(M))\,dt = xyz\int_0^1 3t^2\,dt + C = xyz + C.$$

Thus,

$$\varphi(M) = xyz + C.$$

Using formula (10), find the potentials of the following vector fields:

234. $\mathbf{a} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$, where α, β, γ are constants.

235. $\mathbf{a} = (y + z)\,\mathbf{i} + (x + z)\,\mathbf{j} + (y + x)\,\mathbf{k}$.

236. $\mathbf{a} = y\mathbf{i} + x\mathbf{j} + e^z\mathbf{k}$.

237. $\mathbf{a} = e^x \sin y \cdot \mathbf{i} + e^x \cos y \cdot \mathbf{j} + \mathbf{k}$.

CHAPTER V

THE HAMILTONIAN OPERATOR. SECOND-ORDER DIFFERENTIAL OPERATIONS. THE LAPLACE OPERATOR

Sec. 20. The Hamiltonian operator del

Many operations of vector analysis may be written in abbreviated form and in a form convenient for calculations; this is done through the use of a symbolic operator called the *Hamiltonian operator del*:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{1}$$

This operator combines both differential and vectorial properties. We will regard the formal multiplication of $\partial/\partial x$ by the function $f(x, y, z)$ as the partial differentiation $\partial f/\partial x$.

Within the framework of vector algebra, we will perform the formal operations involving del as if it were a vector. Using this formalism, we obtain the following.

1. If $u = u(x, y, z)$ is a scalar differentiable function, then, by the rule of multiplying a vector by a scalar, we have

$$\nabla u = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) u$$
$$= \mathbf{i}\frac{\partial u}{\partial x} + \mathbf{j}\frac{\partial u}{\partial y} + \mathbf{k}\frac{\partial u}{\partial z} = \operatorname{grad} u. \tag{2}$$

2. If $\mathbf{a} = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$, where P, Q, R are differentiable functions, then, by

the familiar formula for a scalar product, we have

$$(\nabla, \mathbf{a}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z},\ P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}\right)$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \operatorname{div}\mathbf{a}; \quad (3)$$

in particular, $(\nabla, \mathbf{c}) = 0$, where $\mathbf{c}$ is a constant vector.

3. If $\mathbf{a} = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$, then

$$[\nabla, \mathbf{a}] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \operatorname{curl}\ \mathbf{a}; \quad (4)$$

in particular $(\nabla, \mathbf{c}) = 0$, where $\mathbf{c}$ is a constant vector.

Continuing the formalism of operations with ∇ as a vector, we obtain the following from the distributive property of scalar and vector products:

$$(\nabla,\ \mathbf{a} + \mathbf{b}) = (\nabla,\ \mathbf{a}) + (\nabla,\ \mathbf{b}), \quad (5)$$

that is, $\operatorname{div}(\mathbf{a} + \mathbf{b}) = \operatorname{div}\mathbf{a} + \operatorname{div}\mathbf{b}$

$$[\nabla,\ \mathbf{a} + \mathbf{b}] = [\nabla,\ \mathbf{a}] + [\nabla,\ \mathbf{b}] \quad (6)$$

or $\operatorname{curl}(\mathbf{a} + \mathbf{b}) = \operatorname{curl}\mathbf{a} + \operatorname{curl}\mathbf{b}$.

Formulas (5) and (6) may also be interpreted as an exhibition of differential properties of the del operator (∇ is a linear differential operator).

When using the formalism of operations involving the del operator regarded as a vector, one must bear in mind that del is not a vector, for it has neither magnitude nor direction, so that, for example, the vector $[\nabla,\ \mathbf{a}]$ will not, generally speaking, be perpendicular to the vector $\mathbf{a}$ (however, for the plane field $\mathbf{a} = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ the vector

$$[\nabla, \mathbf{a}] = \operatorname{curl}\ \mathbf{a} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

will be perpendicular to the xy-plane and, hence, to the vector $\mathbf{a}$). In the same way, the concept of collinearity is meaningless with respect to the symbolic vector ∇. For example, the expression $[\nabla\varphi,\ \nabla\psi]$, where φ and ψ are

scalar functions, formally resembles a vector product of two collinear vectors, which product is always equal to zero. But this is not true in the general case. Indeed, the vector $\nabla\varphi = \text{grad}\ \varphi$ is directed along the normal to the level surface $\varphi = \text{constant}$, while the vector $\nabla\psi = \text{grad}\ \psi$ defines the normal to the level surface $\psi = \text{constant}$,

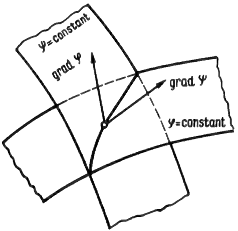

Fig. 35

and in the general case these normals are not necessarily collinear (Fig. 35). On the other hand, in any differentiable scalar field φ we have $[\nabla\varphi, \nabla\varphi] = 0$. These examples show that the del operator must be handled with care.

Besides its vectorial nature, the Hamiltonian operator del has a differential aspect to it. Taking into account the differential aspect of ∇, we will agree that the operator ∇ acts on all quantities that follow it. In this sense, $(\nabla, \mathbf{a}) \neq (\mathbf{a}, \nabla)$. Indeed,

$$(\nabla, \mathbf{a}) = \text{div}\ \mathbf{a},$$

whereas

$$(\mathbf{a}, \nabla) = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$$

is a scalar differential operator.

When applying the del operator to a product of any quantities, one must bear in mind the rule for differentiating a product:

$$\frac{\partial}{\partial x}(uv) = v\,\frac{\partial u}{\partial x} + u\,\frac{\partial u}{\partial x}$$

From this it follows that the del operator must be applied in succession to each factor, leaving the other factors unchanged, and then the sum of the resulting expressions taken. In this procedure we are guided by the following rules.

1°. If the del operator acts on some product, first take into account its differential nature and only then its vectorial properties.

2°. In order to note the fact that the del operator does not act on some quantity involved in a complicated formula, that quantity is labelled with the subscript c (constant), which may be removed in the final result.

3°. All quantities not acted upon by the del operator are placed in front of the operator (that is, to the left) in the final result.

Example 1. Show that

$$\operatorname{div}(u\mathbf{a}) = u \operatorname{div} \mathbf{a} + (\mathbf{a}, \operatorname{grad} u).$$

Here, u is a scalar function and $\mathbf{a}$ is a vector function.

Solution. In symbolic notation,

$$\operatorname{div}(u\mathbf{a}) = (\nabla, u\mathbf{a}).$$

Taking into account first the differential nature of ∇, we have to write

$$(\nabla, u\mathbf{a}) = (\nabla, u_c\mathbf{a}) + (\nabla, u\mathbf{a}_c).$$

Regarding the expression $(\nabla, u_c\mathbf{a})$, we can take the constant factor u_c outside the del sign and, as a scalar, outside the sign of the scalar product; this yields

$$(\nabla, u_c\mathbf{a}) = (u_c\nabla, \mathbf{a}) = u_c(\nabla, \mathbf{a}) = u(\nabla, \mathbf{a}).$$

(The last step is to drop the subscript c).

In the expression $(\nabla, u\mathbf{a}_c)$, the del operator acts only on the scalar function u, and so we can write

$$(\nabla, u\mathbf{a}_c) = (\nabla u, \mathbf{a}_c) = (\mathbf{a}_c, \nabla u) = (\mathbf{a}, \nabla u)$$

to get the formula

$$(\nabla,\ u\mathbf{a}) = u\,(\nabla,\ \mathbf{a}) + (\mathbf{a},\ \nabla u)$$

or

$$\operatorname{div}(u\mathbf{a}) = u \operatorname{div}\mathbf{a} + (\mathbf{a},\ \operatorname{grad} u).$$

Example 2. Show that

$$\operatorname{curl}(u\mathbf{a}) = u\operatorname{curl}\mathbf{a} - [\mathbf{a},\ \operatorname{grad} u].$$

Solution. In symbolic notation,

$$\operatorname{curl}(u\mathbf{a}) = (\nabla,\ u\mathbf{a}).$$

Taking into account the differential properties of ∇, we first write

$$[\nabla,\ u\mathbf{a}] = [\nabla,\ u_c\mathbf{a}] + [\nabla,\ u\mathbf{a}_c]. \tag{7}$$

Then in the first term on the right we take the scalar factor u_c outside the del sign and outside the sign of the vector product, which yields

$$[\nabla,\ u_c\mathbf{a}] = u_c\,[\nabla,\ \mathbf{a}] = u\,[\nabla,\ \mathbf{a}].$$

In the second term in (7) we refer u to the operator ∇ and change the order of the factors so that the vector $\mathbf{a}_c$, which del does not act on, is in front of ∇. This yields

$$[\nabla,\ u\mathbf{a}_c] = [\nabla u,\ \mathbf{a}_c] = -[\mathbf{a},\ \nabla u].$$

Thus

$$[\nabla,\ u\mathbf{a}] = u\,[\nabla,\ \mathbf{a}] - [\mathbf{a},\ \nabla u]$$

or

$$\operatorname{curl}(u\mathbf{a}] = u\operatorname{curl}\mathbf{a} - [\mathbf{a},\ \operatorname{grad} u].$$

Example 3. Use the symbolic method to find div $[\mathbf{a},\ \mathbf{b}]$

Solution. We have

$$\operatorname{div}[\mathbf{a},\ \mathbf{b}] = (\nabla,\ [\mathbf{a},\ \mathbf{b}]) = (\nabla,\ [\mathbf{a},\ \mathbf{b}_c]) + (\nabla,\ [\mathbf{a}_c,\ \mathbf{b}]). \tag{8}$$

Using the property of cyclic permutation of factors in a mixed product, we transform the expression on the right of (8) so that all constant quantities are in front of the del operator and all variable quantities follow it. This yields

$$\operatorname{div}[\mathbf{a},\ \mathbf{b}] = ([\nabla,\ \mathbf{a}],\ \mathbf{b}_c) - (\nabla,\ [\mathbf{b},\ \mathbf{a}_c])$$

$$= ([\nabla,\ \mathbf{a}],\ \mathbf{b}_c) - ([\nabla,\ \mathbf{b}],\ \mathbf{a}_c) = (\mathbf{b},\ [\nabla,\ \mathbf{a}]) - (\mathbf{a},\ [\nabla,\ \mathbf{b}])$$

or $\operatorname{div}[\mathbf{a},\ \mathbf{b}] = (\mathbf{b},\ \operatorname{curl}\mathbf{a}) - (\mathbf{a},\ \operatorname{curl}\mathbf{b})$.

Remark. The use of the symbolic method enables us to avoid cumbersome analytical transformations and obtain the final results very quickly. On the other hand, the various formal transformations involving the del operator must be performed with extreme caution in order to avoid serious mistakes. For this reason, if there is any doubt about the final result, it is wise to verify it by the analytical method.

238. Show that

(a) $$\nabla\left(\frac{u}{v}\right)=\frac{v\nabla u-u\nabla v}{v^2};$$

(b) $$\nabla f(u) = f'(u)\,\nabla u.$$

239. Prove that the vector $[\nabla u, \nabla v]$ is solenoidal if u and v are differentiable scalar functions.

Use the Hamiltonian operator del (∇) to prove the following equations:

240. (a) $\operatorname{grad}(uv) = v \operatorname{grad} u + u \operatorname{grad} v$;

(b) $\operatorname{curl}(\mathbf{a}, \mathbf{b}) = (\mathbf{b}, \nabla)\,\mathbf{a} - (\mathbf{a}, \nabla)\,\mathbf{b} + \mathbf{a} \operatorname{div} \mathbf{b} - \mathbf{b} \operatorname{div} \mathbf{a}$.

241. $\operatorname{curl}[\mathbf{a}, \mathbf{r}] = 2\mathbf{a}$, where $\mathbf{a}$ is a constant vector.

242. Prove that the vector $\mathbf{a} = u \operatorname{grad} v$ is orthogonal to $\operatorname{curl} \mathbf{a}$.

Sec. 21. Second-order differential operations. The Laplace operator

Second-order differential operations are obtained as a result of a twofold application of the operator ∇ to fields.

Suppose we have a scalar field $u = u(M)$. In this field, the operator ∇ generates a vector field $\nabla u = \operatorname{grad} u$.

In the vector field ∇u, the operator ∇, when applied a second time to ∇u, yields the scalar field

$$(\nabla, \nabla u) = \operatorname{div} \operatorname{grad} u \tag{1}$$

and the vector field

$$(\nabla, \nabla u) = \operatorname{curl} \operatorname{grad} u. \tag{2}$$

If a vector field $\mathbf{a} = \mathbf{a}(M)$ is given, the operator ∇ generates in it a scalar field $(\nabla, \mathbf{a}) = \operatorname{div} \mathbf{a}$. In the scalar

ield div **a** the operator ∇ generates a vector field

$$\nabla(\nabla, \mathbf{a}) = \text{grad div } \mathbf{a}. \qquad (3)$$

In the vector field $\mathbf{a} = \mathbf{a}(M)$, the operator ∇ also generates a vector field $[\nabla, \mathbf{a}] = \text{curl } \mathbf{a}$. If the operator ∇ is again applied to this field, we obtain the scalar field

$$(\nabla, [\nabla, \mathbf{a}]) = \text{div curl } \mathbf{a} \qquad (4)$$

and the vector field

$$[\nabla, [\nabla, \mathbf{a}]] = \text{curl curl } \mathbf{a}. \qquad (5)$$

The formulas (1) to (5) define what are called *differential operations of the second order*.

Example 1. Suppose a function $u = u(x, y, z)$ has continuous partial derivatives up to second order inclusive. Prove that

$$\text{curl grad } u \equiv 0.$$

Solution. 1st method. Operating formally, we obtain

$$\text{curl grad } u = [\nabla, \nabla u] = [\nabla, \nabla] u = 0$$

since $[\nabla, \nabla] = 0$ being the vector product of two identical "vectors".

2nd method. Using the expressions for the gradient and curl in Cartesian coordinates and taking into account the given conditions, we have

$$\text{curl grad } u = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{vmatrix} = \left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right)\right]\mathbf{i}$$

$$+\left[\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\right]\mathbf{k}$$

$$= \left(\frac{\partial^2 u}{\partial y\,\partial z} - \frac{\partial^2 u}{\partial z\,\partial y}\right)\mathbf{i} + \left(\frac{\partial^2 u}{\partial z\,\partial x} - \frac{\partial^2 u}{\partial x\,\partial z}\right)\mathbf{j} + \left(\frac{\partial^2 u}{\partial x\,\partial y} - \frac{\partial^2 u}{\partial y\,\partial x}\right)\mathbf{k} = 0,$$

since the mixed derivatives are equal in this case,

$$\frac{\partial^2 u}{\partial y\,\partial z} = \frac{\partial^2 u}{\partial z\,\partial y}, \quad \frac{\partial^2 u}{\partial z\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial z}, \quad \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}.$$

In similar fashion proof is given that for the vector field

$$\mathbf{a} = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k},$$

the coordinates P, Q, R of which have continuous partial derivatives of the second order, we obtain div curl $\mathbf{a} = 0$.

Note particularly the second-order differential operation div grad $u = (\nabla, \nabla u)$. Assuming that the function $u(x, y, z)$ has second partial derivatives with respect to x, y, z, we obtain

$$(\nabla, \nabla u) = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k},\ \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \equiv \Delta u.$$

Thus, $(\nabla, \nabla u) = \Delta u$, where the symbol

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

is termed the *Laplace operator* (or *Laplacian*). It may be represented as a scalar product of the Hamiltonian operator ∇ into itself, that is,

$$\Delta = (\nabla, \nabla) = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

This operator plays an important role in mathematical physics.

Let us examine another second-order operation curl curl $\mathbf{a}$. We have curl curl $\mathbf{a} = [\nabla, [\nabla, \mathbf{a}]]$. Let us take advantage of the formula for a vector triple product written as $[\mathbf{A}, [\mathbf{B}, \mathbf{C}]] = \mathbf{B}(\mathbf{A}, \mathbf{C}) - (\mathbf{A}, \mathbf{B})\,\mathbf{C}$. Replacing $\mathbf{A}$ by ∇, $\mathbf{B}$ by ∇, and $\mathbf{C}$ by $\mathbf{a}$, we obtain

$$[\nabla, [\nabla, \mathbf{a}]] = \nabla(\nabla, \mathbf{a}) - (\nabla, \nabla)\,\mathbf{a} = \nabla(\nabla, \mathbf{a}) - \Delta\mathbf{a} \quad (6)$$

or

curl (curl $\mathbf{a}$) = grad div $\mathbf{a} - \Delta\mathbf{a}$, where $\Delta\mathbf{a} =$

$= \Delta P\cdot\mathbf{i} + \Delta Q\cdot\mathbf{j} + \Delta R\cdot\mathbf{k}$.

The following table is a pictorial display of second-order differential operations:

	Scalar field u	Vector field a	
	grad	div	curl
grad		grad div a	
div	div grad $u = \Delta u$		div curl $\mathbf{a} = 0$
curl	curl grad $u = 0$		curl curl a $= \text{grad div } \mathbf{a} - \Delta \mathbf{a}$

Example 2. The laws of the classical theorem of electromagnetism are postulated in the form of a system of Maxwell equation.

In the most elementary case of a nonconducting homogeneous and isotropic medium and in the absence of charges and currents, this system is of the form

$$\frac{\varepsilon}{c}\frac{\partial \mathbf{E}}{\partial t} = [\nabla,\ \mathbf{H}]. \tag{7}$$

$$-\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t} = [\nabla,\ \mathbf{E}], \tag{8}$$

$$(\nabla,\ \mathbf{E}) = 0, \tag{9}$$

$$(\nabla,\ \mathbf{H}) = 0. \tag{10}$$

Here, E and H are vectors of the electric-field and magnetic-field intensity; ε and μ are coefficients of the permittivity and permeability (our assumptions are that ε and μ are constants); c is the velocity of light in empty space.

Since the spatial and time derivatives commute, that is,

$$\frac{\partial}{\partial t}[\nabla,\ \mathbf{H}] = \left[\nabla,\ \frac{\partial \mathbf{H}}{\partial t}\right],$$

it follows that by differentiating (7) with respect to t, we obtain

$$\frac{\varepsilon}{c}\frac{\partial^2 \mathbf{E}}{\partial t^2}=\left[\nabla,\ \frac{\partial \mathbf{H}}{\partial t}\right].$$

Replacing $\frac{\partial \mathbf{H}}{\partial t}$ taken from (8), we get $\frac{\varepsilon}{c}\frac{\partial^2 \mathbf{E}}{\partial t^2}=$ $=-\frac{c}{\mu}[\nabla,\ [\nabla,\ \mathbf{E}]]$ or

$$\frac{\varepsilon\mu}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}=[\nabla,\ [\nabla,\ \mathbf{E}]]. \tag{11}$$

By virtue of formula (6), $[\nabla,\ [\nabla,\ \mathbf{E}]]=\nabla(\nabla,\ \mathbf{E})-\Delta\mathbf{E}$. Since $(\nabla,\ \mathbf{E})=0$, it follows from (11) that $\frac{\varepsilon\mu}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}=\Delta\mathbf{E}$.

To summarize, for the vector field $\mathbf{E}$ we obtain the equation

$$\frac{\partial^2 \mathbf{E}}{\partial t^2}=\frac{c^2}{\varepsilon\mu}\Delta\mathbf{E}.$$

This is one of the basic equations of mathematical physics and is called the *wave equation*.

It is easy to see (check this!) that the vector field $\mathbf{H}$ satisfies the same kind of wave equation

$$\frac{\partial^2 \mathbf{H}}{\partial t^2}=\frac{c^2}{\varepsilon\mu}\Delta\mathbf{H}.$$

Thus, under our conditions, each of the coordinates E_x, E_y, E_z and H_x, H_y, H_z of the vectors $\mathbf{E}$ and $\mathbf{H}$ satisfies the equation

$$\frac{\partial^2 u}{\partial t^2}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right).$$

Here, $a=c/\sqrt{\varepsilon\mu}$ is the velocity of propagation of the process. In a vacuum, where $\varepsilon=\mu=1$, we have $a=c$, that is to say, in a vacuum, electromagnetic processes are propagated with the velocity of light.

243. Show that any solution of the equation $[\nabla,\ [\nabla,\ \mathbf{A}]]-$ $-k^2\mathbf{A}=0$ that satisfies the solenoidal condition satisfies the Helmholtz vector equation

$$\nabla^2\mathbf{A}+k^2\mathbf{A}=0.$$

Definition. The scalar field $u = u(x, y, z)$ that satisfies the condition $\Delta u = 0$ is termed a *Laplace* (or *harmonic*) *field*.

Example 3. An important instance of a harmonic field is the scalar field $u=k/r$, $k=$constant, $r = \sqrt{x^2+y^2+z^2}$. This function is the potential of a gravitational field generated by a point mass placed at the coordinate origin. It is easy to verify that the function $u = k/r$ is harmonic everywhere except at the coordinate origin, where it is not defined. Indeed,

$$\left(\nabla, \nabla \frac{k}{r}\right) = k\left(\nabla, \nabla\frac{1}{r}\right) = k\left(\nabla, -\frac{1}{r^2}\nabla r\right)$$
$$= -k\left(\nabla, \frac{1}{r^2}\mathbf{r}^0\right) = -k\left(\nabla\frac{1}{r^2}, \mathbf{r}^0\right) - k\frac{1}{r^2}(\nabla, \mathbf{r}^0)$$
$$= -k\left(-\frac{2}{r^3}\nabla r, \mathbf{r}^0\right) - \frac{k}{r^2}(\nabla, \mathbf{r}^0)$$
$$= \frac{2k}{r^3}(\mathbf{r}^0, \mathbf{r}^0) - \frac{k}{r^2}(\nabla, \mathbf{r}^0) = \frac{2k}{r^3} - \frac{k}{r^2}\cdot\frac{2}{r} = 0$$

for all $r \neq 0$ since

$$(\nabla, \mathbf{r}^0) = \left(\nabla, \frac{\mathbf{r}}{r}\right) = \left(\nabla\frac{1}{r}, \mathbf{r}\right) + \frac{1}{r}(\nabla, \mathbf{r})$$
$$= \left(-\frac{\nabla r}{r^2}, \mathbf{r}\right) + \frac{3}{r} = -\frac{1}{r^2}(\mathbf{r}^0, \mathbf{r}) + \frac{3}{r}$$
$$= -\frac{1}{r}(\mathbf{r}^0, \mathbf{r}^0) + \frac{3}{r} = -\frac{1}{r} + \frac{3}{r} = \frac{2}{r}.$$

Example 4. Prove that in the potential field of a vector $\mathbf{a}$ its potential function $u(x, y, z)$ satisfies the Poisson equation

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \rho(x, y, z), \qquad (12)$$

where $\rho(x, y, z)$ is the divergence of the vector $\mathbf{a}$.

Solution. It is given that

$$\operatorname{div} \mathbf{a} = \rho. \qquad (13)$$

Since the field of the vector $\mathbf{a}$ is a potential field, it follows that $\mathbf{a} = \operatorname{grad} u$, where u is the potential of the field. Substituting $\mathbf{a} = \operatorname{grad} u$ into 13), we obtain

div grad $u = \rho$ or, since div grad $u = \Delta u$, we have $\Delta u = \rho$.

In the special case of points of the field where the divergence is zero, equation (12) turns into the Laplace equation $\Delta u = 0$. The Laplace-Poisson equation permits finding the potential function u by integrating the partial

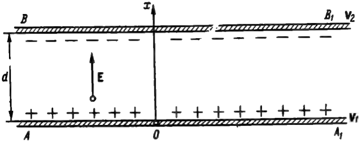

Fig. 36

differential equation. This turns out to be more convenient in some problems.

In electrostatics, preference is often given to finding the function $v = -u$ instead of u. Then $\mathbf{a} = -\text{grad } v$. Accordingly, in the theory of the electrostatic field, the Poisson equation is of the form

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = -\frac{4\pi\rho}{\varepsilon}. \qquad (14)$$

Let us consider an elementary case where the Poisson equation is employed.

Example 5. Suppose two infinite parallel plates AA_1 and BB_1 with opposite charges have potentials v_1 and v_2; for the sake of definiteness, $v_1 > v_2$. Find the field $\mathbf{E}$ between them (Fig. 36).

Solution. Send the x-axis at right angles to the plates in the direction of decreasing potential, and bring the yz-plane to coincidence with the positively charged plate AA_1. We now seek the potential function from the Poisson equation. By virtue of the symmetry of the problem about the x-axis and due to the infinity of the plates, we can conclude that the equipotential surfaces are planes

parallel to the plates, and the function v depends solely on the variable x. Equation (14) takes the form

$$\frac{d^2v}{dx^2}=0 \tag{15}$$

since space charges are absent throughout the space between the plates. Integrating (15), we find

$$v = C_1x + C_2, \tag{16}$$

where C_1 and C_2 are arbitrary constants.

We require that for $x = 0$ the function v take the value v_1 and for $x = d$, where d is the distance between the plates, that it take the value v_2. This yields $C_2=v_1$, $v_2= C_1d + C_2$, whence $C_2 = v_1$, $C_1 = (v_2 - v_1)/d$. Substituting these values of C_1 and C_2 into (16), we obtain

$$v=\frac{v_2-v_1}{d}x+v_1=v_1-\frac{v_1-v_2}{d}x.$$

The vector $\mathbf{E}$ is found from the formula $\mathbf{E} = -\operatorname{grad} v$, which yields

$$\mathbf{E}=\frac{v_1-v_2}{d}\,\mathbf{i}$$

so that the field is homogeneous and in the direction of the x-axis. The magnitude of $\mathbf{E}$ is equal at every point to $|\mathbf{E}| = (v_1 - v_2)/d$, that is, it is equal to the potential drop per unit length of the shortest distance between the plates.

244. Suppose a scalar function $\varphi(M)$ satisfies the Laplace equation. Show that the vector $\nabla\varphi$ is solenoidal and irrotational.

245. Show that $\Delta(uv) = u\,\Delta v + v\,\Delta u + 2(\nabla u, \nabla v)$.

246. Prove that if $\mathbf{r}$ is the radius vector, then

$$\Delta r=\begin{cases}\dfrac{2}{r} & \text{in space,}\\[2mm] \dfrac{1}{r} & \text{in the plane.}\end{cases}$$

247. Check to see whether the following scalar fields are harmonic or not:

(a) $u = x^2 + 2xy - y^2$,

(b) $u = x^2y + y^2z + z^2x$,

(c) $u = x^2 - y^2$.

248. Show that the scalar field

$$u = \ln \frac{1}{r}, \quad \text{where } r = \sqrt{x^2 + y^2} \quad (r \neq 0),$$

is harmonic.
249. Find all harmonic fields that depend solely on x.
250. Find the general form of a homogeneous harmonic polynomial of second degree in x and y.
251. Find all solutions of the Poisson equation $\Delta u = x^{n-2}$ that depend solely on x.

Example 6. *Green's formulas.* Let φ, ψ be two scalar functions of a point. Set up the vector $\mathbf{a} = \varphi \operatorname{grad} \psi$. Then

$$\operatorname{div} \mathbf{a} = \operatorname{div} (\varphi \operatorname{grad} \psi) = \varphi \operatorname{div} \operatorname{grad} \psi + (\operatorname{grad} \varphi, \operatorname{grad} \psi) = \varphi \cdot \Delta\psi + (\operatorname{grad} \varphi, \operatorname{grad} \psi).$$

Now apply the Gauss-Ostrogradsky formula

$$\oint\!\!\oint_{\Sigma} (\mathbf{a}, \mathbf{n}^0)\, d\sigma = \iiint_V \operatorname{div} \mathbf{a}\, dv.$$

Note that in our case

$$(\mathbf{a}, \mathbf{n}^0) = (\varphi \operatorname{grad} \psi, \mathbf{n}^0) = \varphi (\operatorname{grad} \psi, \mathbf{n}^0) = \varphi \frac{\partial \psi}{\partial n}.$$

We thus obtain *Green's first formula:*

$$\iiint_V [\varphi \Delta\psi + (\operatorname{grad} \varphi, \operatorname{grad} \psi)]\, dv = \oint\!\!\oint_{\Sigma} \varphi \frac{\partial \psi}{\partial n}\, d\sigma, \quad (17)$$

which for $\varphi = \psi$ turns into

$$\iiint_V [\varphi \Delta\varphi + |\operatorname{grad} \varphi|^2]\, dv = \oint\!\!\oint_{\Sigma} \varphi \frac{\partial \varphi}{\partial n}\, d\sigma. \quad (18)$$

If we put $\varphi \equiv 1$ in (17) we get

$$\iiint_V \Delta\psi\, dv = \oint\!\!\oint \frac{\partial \psi}{\partial n}\, d\sigma.$$

In (17), interchange φ and ψ and subtract the resulting formula

$$\iiint_V [\psi \Delta\varphi + (\operatorname{grad} \psi, \operatorname{grad} \varphi)]\, dv = \oint\!\!\oint_{\Sigma} \psi \frac{\partial \varphi}{\partial n}\, d\sigma$$

from (17). This then yields *Green's second formula*:

$$\iiint_V (\varphi\cdot\Delta\psi-\psi\cdot\Delta\varphi)\,dv=\oiint_\Sigma\left(\varphi\frac{d\psi}{\partial n}-\psi\frac{\partial\varphi}{\partial n}\right)d\sigma.$$

It is assumed here that all functions that we have to deal with and also all their derivatives that occur in the formulas are continuous in the region under consideration.

Example 7. Find the surface integral

$$I=\oiint_\Sigma \varphi\frac{\partial\psi}{\partial n}\,d\sigma$$

taken over the sphere Σ: $x^2+y^2+z^2=1$ for $\varphi = x^2+y^2$ and $\psi=y^2+z^2$.

Solution. By Green's first formula, the desired integral is

$$I=\iiint_V [\varphi\Delta\psi+(\operatorname{grad}\varphi,\ \operatorname{grad}\psi)]\,dv,$$

where the region of integration V is a sphere: $x^2+y^2+z^2\leqslant 1$.

We have $\Delta\psi=4$, $\operatorname{grad}\varphi=2x\mathbf{i}+2y\mathbf{j}$, $\operatorname{grad}\psi=2y\mathbf{j}+2z\mathbf{k}$, $(\operatorname{grad}\varphi,\ \operatorname{grad}\psi)=4y^2$ and therefore

$$I=\iiint_V (4x^2+4y^2+4y^2)\,dv=4\iiint_V (x^2+2y^2)\,dv.$$

Passing to the spherical coordinates $x=r\cos\varphi\sin\theta$, $y=r\sin\varphi\sin\theta$, $z=r\cos\theta$, we obtain

$$\begin{aligned}I&=4\iiint_V (r^2\cos^2\varphi\sin^2\theta+2r^2\sin^2\varphi\sin^2\theta)\,r^2\sin\theta\,dr\,d\theta\,d\varphi\\&=4\int_0^{2\pi}(\cos^2\varphi+2\sin^2\varphi)\,d\varphi\int_0^{\pi}\sin^3\theta\,d\theta\int_0^1 r^4\,dr\\&=\frac{4}{5}\int_0^{2\pi}(1+\sin^2\varphi)\,d\varphi\int_0^{\pi}(1-\cos^2\theta)\sin\theta\,d\theta\\&=\frac{12}{5}\pi\left(-\cos\theta+\frac{1}{3}\cos^3\theta\right)\Big|_0^{\pi}=\frac{16}{5}\pi.\end{aligned}$$

Example 8. Find the surface integral

$$I = \oint\!\!\oint_{\Sigma} \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n} \right) d\sigma$$

taken over the surface Σ: $x^2 + y^2 = R^2$, $z = 0$, $z = H$, $H > 0$, provided $\varphi = x^2 + y^2 + x + z$, $\psi = x^2 + y^2 + 2z + x$.

Solution. By Green's second formula, the desired integral is

$$I = \iiint_V (\varphi \Delta \psi - \psi \Delta \varphi)\, dv.$$

For the given functions φ and ψ we have $\Delta\varphi = 4$, $\Delta\psi = 4$ and therefore

$$I = -4 \iiint_V z\, dv.$$

Passing to the cylindrical coordinates $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, $z = z$, we obtain

$$I = -4 \iiint_V z\rho\, d\rho\, d\varphi\, dz = -4 \int_0^{2\pi} d\varphi \int_0^R \rho\, d\rho \int_0^H z\, dz = -2\pi R^2 H^2.$$

Example 9. Find the surface integral

$$I = \oint\!\!\oint_{\Sigma} \varphi \frac{\partial \varphi}{\partial n}\, d\sigma$$

over a closed surface Σ bounded by the planes $x + y + z = 1$, $x = 0$, $y = 0$, $z = 0$, provided $\varphi = e^x \sin y + 1$.

Solution. The given function is harmonic since $\Delta\varphi = e^x \sin y - e^x \sin y \equiv 0$. Therefore, by (18) we have

$$I = \iiint_V |\operatorname{grad} \varphi|^2\, dv.$$

We find the modulus of the gradient of the function φ:

$$\operatorname{grad} \varphi = e^x \sin y \cdot \mathbf{i} + e^x \cos y \cdot \mathbf{j}, \quad |\operatorname{grad} \varphi| = e^x.$$

The desired integral is equal to

$$I = \iiint_V e^{2x}\, dv = \int_0^1 e^{2x}\, dx \int_0^{1-x} dy \int_0^{1-x-y} dz = \frac{1}{8}(e^2 - 5).$$

252. Compute the surface integral $I=\oint\!\!\oint_{\Sigma} \varphi \frac{\partial \psi}{\partial n}\, d\sigma$ over a closed surface Σ: $\{x^2+y^2+z^2=1,\ y=0,\ y\geqslant 0\}$, provided $\varphi=z^2$, $\psi=x^2+y^2-z^2$.

253. Compute the surface integral $I=\oint\!\!\oint_{\Sigma} \left(\varphi \frac{\partial \psi}{\partial n}-\psi \frac{\partial \varphi}{\partial n}\right) d\sigma$ taken over the entire surface of a closed cylinder Σ: $\{x^2+y^2=1,\ z=0,\ z=1\}$, provided $\varphi=2x^2$, $\psi=x^2+z^2$.

254. Compute the surface integral $I=\oint\!\!\oint_{\Sigma} \varphi \frac{\partial \varphi}{\partial n}\, d\sigma$, provided $\varphi=(x+y+z)/\sqrt{3}$ and Σ is a sphere: $x^2+y^2+z^2=R^2$.

255. Find the surface integral $I=\oint\!\!\oint_{\Sigma} \frac{\partial \varphi}{\partial n}\, d\sigma$, provided $\varphi=e^x \sin y+e^y \sin x+z$ and Σ is a triaxial ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$.

Sec. 22. Vector potential

Let a vector field

$$\mathbf{a} = \mathbf{a}\,(M) = P\,(x, y, z)\,\mathbf{i} + Q\,(x, y, z)\,\mathbf{j} + R\,(x, y, z)\,\mathbf{k}$$

be solenoidal in the region G, that is, $\operatorname{div} \mathbf{a}\,(M) = 0$ in G.

Definition. The *vector potential* of a vector field $\mathbf{a} = \mathbf{a}\,(M)$ is a vector $\mathbf{b}\,(M)=P_1\,(x, y, z)\,\mathbf{i} + Q_1\,(x, y, z)\,\mathbf{j} + R_1\,(x, y, z)\,\mathbf{k}$ that satisfies in G the condition

$$\operatorname{curl} \mathbf{b}\,(M) = \mathbf{a}\,(M) \tag{1}$$

or, in coordinate form,

$$\frac{\partial R_1}{\partial y}-\frac{\partial Q_1}{\partial z}=P,\quad \frac{\partial P_1}{\partial z}-\frac{\partial R_1}{\partial x}=Q,\quad \frac{\partial Q_1}{\partial x}-\frac{\partial P_1}{\partial y}=R. \tag{2}$$

For a solenoidal vector field $\mathbf{a}\,(M)$ the vector potential $\mathbf{b}\,(M)$ is not defined uniquely: the condition (1) is also satisfied by the vector $\mathbf{B}\,(M) = \mathbf{b}\,(M) + \operatorname{grad} f\,(M)$, where $f\,(M)$ is an arbitrary differentiable scalar function, since $\operatorname{curl}\,(\operatorname{grad} f\,(M)) \equiv 0$.

Thus, two vector potentials of the solenoidal field $\mathbf{a}\,(M)$ differ by the gradient of the scalar field.

Finding the vector potential $\mathbf{b}(M)$ of the solenoidal field $\mathbf{a}(M)$ reduces to finding some particular solution of system (2) of three partial differential equations for the three unknown functions $P_1(x, y, z)$, $Q_1(x, y, z)$, $R_1(x, y, z)$.

The vector potential $\mathbf{b}(M)$ may be constructed in the following manner. Taking advantage of the arbitrariness of choice of the vector $\mathbf{b}(M)$, we will simplify matters by setting $P_1(x, y, z) \equiv 0$, that is, the vector $\mathbf{b}(M)$ will be sought in the form $\mathbf{b}(M) = Q_1(x, y, z)\,\mathbf{j} + R_1(x, y, z)\,\mathbf{k}$. Then the system of differential equations (2) for finding the unknown functions $Q_1(x, y, z)$ and $R_1(x, y, z)$ takes the form

$$\left.\begin{aligned} \frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z} &= P, \\ \frac{\partial R_1}{\partial x} &= -Q, \\ \frac{\partial Q_1}{\partial x} &= R. \end{aligned}\right\} \qquad (3)$$

From the second and third equations of this system we find

$$R_1(x, y, z) = -\int Q(x, y, z)\,dx + C_1(y, z),$$

$$Q_1(x, y, z) = \int R(x, y, z)\,dx + C_2(y, z),$$

where $C_1(y, z)$ and $C_2(y, z)$ are any differentiable functions of y and z. For the sake of simplicity, set $C_2(y, z) \equiv 0$ and choose the function $C_1(y, z)$ so that the first equation of system (3) is satisfied. To do this, we substitute into the first equation the expressions that were found for Q_1 and R_1:

$$-\frac{\partial}{\partial y}\int Q(x, y, z)\,dx + \frac{\partial C_1}{\partial y} - \frac{\partial}{\partial z}\int R(x, y, z)\,dx = P(x, y, z).$$

From this we get

$$\frac{\partial C_1}{\partial y} = \frac{\partial}{\partial y}\int Q(x, y, z)\,dx + \frac{\partial}{\partial z}\int R(x, y, z)\,dx + P(x, y, z).$$

It is easy to verify that the right-hand side of this equation does not depend on x; this is because $\operatorname{div} \mathbf{a}(M) = 0$ in G.

Integrating the last equation with respect to y, we find

$$C_1(y, z) = \int \left[\frac{\partial}{\partial y} \int Q(x, y, z)\, dx + \frac{\partial}{\partial z} \int R(x, y, z)\, dx + P(x, y, z) \right] dy + C_3(z). \quad (4)$$

Setting $C_3(z) \equiv 0$ in (4) and substituting (4) into the expression for $R_1(x, y, z)$, we get a particular solution of system (3):

$$P_1 \equiv 0, \quad (5)$$

$$Q_1 = \int R(x, y, z)\, dx, \quad (6)$$

$$R_1 = \int \left[\frac{\partial}{\partial y} \int Q(x, y, z)\, dx + \frac{\partial}{\partial z} \int R(x, y, z)\, dx + P(x, y, z) \right] dy - \int Q(x, y, z)\, dx. \quad (7)$$

The vector $\mathbf{b}(M)$, whose coordinates $P_1(x, y, z)$, $Q_1(x, y, z)$, $R_1(x, y, z)$ are defined by formulas (5), (6), (7), is the vector potential since it satisfies the condition $\operatorname{curl} \mathbf{b} = \mathbf{a}$.

Example 1. Find the vector potential $\mathbf{b} = \mathbf{b}(x, y, z)$ for a solenoidal field given by the vector

$$\mathbf{a} = 2y\mathbf{i} - z\mathbf{j} + 2x\mathbf{k}.$$

Solution. We seek the potential $\mathbf{b}$ in the form

$$\mathbf{b} = \mathbf{b}(x, y, z) = Q_1(x, y, z)\,\mathbf{j} + R_1(x, y, z)\,\mathbf{k},$$

where $Q_1(x, y, z)$ and $R_1(x, y, z)$ are found from (7) and (8). Since in the given case $P = 2y$, $Q = -z$, $R = 2x$, we obtain

$$Q_1(x, y, z) = \int 2x\, dx = x^2,$$

$$R_1(x, y, z) = \int z\, dx + \int 2y\, dy = xz + y^2.$$

Thus

$$\mathbf{b}(x, y, z) = x^2\mathbf{j} + (xz + y^2)\,\mathbf{k}.$$

It is clear, by direct verification, that curl $\mathbf{b} = \mathbf{a}$, and, hence, this vector is the vector potential of the given field.

Remark. Due to the arbitrariness in choosing the vector $\mathbf{b}$, we could require $Q_1(x, y, z) \equiv 0$ or $R_1(x, y, z) \equiv 0$ instead of $P_1(x, y, z) \equiv 0$. The system of equations (2) and formulas (5), (6), (7) would naturally change.

Find the vector potentials of the following solenoidal fields:

256. $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

257. $\mathbf{a} = 2y\mathbf{i} + 2z\mathbf{j}$.

258. $\mathbf{a} = (e^x - e^y)\mathbf{k}$.

259. $\mathbf{a} = 6y^2\mathbf{i} + 6z\mathbf{j} + 6x\mathbf{k}$.

260. $\mathbf{a} = 3y^2\mathbf{i} - 3x^2\mathbf{j} - (y^2 + 2x)\mathbf{k}$.

261. $\mathbf{a} = ye^{x^2}\mathbf{i} + 2yz\mathbf{j} - (2xyze^{x^2} + z^2)\mathbf{k}$.

If the vector field $\mathbf{a} = \mathbf{a}(M)$ is solenoidal in the region G, which is star-shaped (see Sec. 19, Chapter IV) with centre at the coordinate origin $O(0, 0, 0)$ [the field $\mathbf{a}(M)$ may not be defined at the point O], then one of the vector potentials $\mathbf{b} = \mathbf{b}(M)$ may be found from the formula

$$\mathbf{b}(M) = \int_0^1 [\mathbf{a}(M'), \mathbf{r}(M)]\, t\, dt, \qquad (8)$$

where $\mathbf{r}(M) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the radius vector of the point $M(x, y, z)$ and the point $M'(tx, ty, tz)$ runs over the line segment OM as the parameter t varies from 0 to 1.

Example 2. Use formula (8) to find the vector potential of the solenoidal field

$$\mathbf{a} = 2y\mathbf{i} - z\mathbf{j} + 2x\mathbf{k}.$$

Solution. The given vector field is defined throughout three-dimensional space, which is a star-shaped region with centre at the coordinate origin, and so we can use (8) to find the vector potential. At the point $M'(tx, ty, tz)$ we have

$$\mathbf{a}(M') = 2ty\mathbf{i} - tz\mathbf{j} + 2tx\mathbf{k}.$$

We find the vector product

$$[\mathbf{a}(M'), \mathbf{r}(M)] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2ty & -tz & 2tx \\ x & y & z \end{vmatrix}$$

$$= [-(2xy+z^2)\,\mathbf{i} + (2x^2-2yz)\,\mathbf{j} + (2y^2+xz)\,\mathbf{k}]\,t.$$

Using (8), we obtain

$$\mathbf{b}(M) = \int_0^1 [-(2xy+z^2)\,\mathbf{i} + (2x^2-2yz)\,\mathbf{j} + (2y^2+xz)\,\mathbf{k}]\,t^2\,dt$$

$$= -\frac{1}{3}(2xy+z^2)\,\mathbf{i} + \frac{2}{3}(x^2-yz)\,\mathbf{j} + \frac{1}{3}(2y^2+xz)\,\mathbf{k}.$$

It is easy to establish that curl $\mathbf{b}(M) = \mathbf{a}(M)$.

Remark. In examples 1 and 2 we obtain different vector potentials for one and the same solenoidal field $\mathbf{a} = 2y\mathbf{i} - z\mathbf{j} + 2x\mathbf{k}$:

$$\mathbf{b}_1(M) = x^2\mathbf{j} + (xz+y^2)\,\mathbf{k},$$

$$\mathbf{b}_2(M) = -\frac{1}{3}(2xy+z^2)\,\mathbf{i} + \frac{2}{3}(x^2-yz)\,\mathbf{j} + \frac{1}{3}(2y^2+xz)\,\mathbf{k}.$$

They differ by a term equal to the gradient of some scalar field $f(M)$. This term plays the role of an arbitrary constant (when acted on by the curl). It may be represented as the gradient of some scalar function $f(M)$. Let us find this function in our example. We have

$$\operatorname{grad} f(M) = \mathbf{b}_1(M) - \mathbf{b}_2(M)$$

$$= \frac{1}{3}(2xy+z^2)\,\mathbf{i} + \frac{1}{3}(x^2+2yz)\,\mathbf{j} + \frac{1}{3}(2xz+y^2)\,\mathbf{k}.$$

To find the scalar field $f(M)$, use formula (3) of Sec. 19, in which we take the coordinate origin $O(0, 0, 0)$ for the point (x_0, y_0, z_0). This yields

$$f(M) = \int_0^x 0\cdot dx + \int_0^y \frac{1}{3}x^2\,dx + \int_0^z \frac{1}{3}(2xz+y^2)\,dz + C$$

$$= \frac{1}{3}(x^2y + y^2z + z^2x) + C,$$

where C is an arbitrary constant.

Example 3. Find the vector potential **b** of a magnetic field **H** set up by an electric charge e that is moving with a constant velocity **v**.

Solution. By the Biot-Savart law, the intensity of the magnetic field is

$$\mathbf{H}(M)=\frac{[e\mathbf{v},\ \mathbf{r}]}{4\pi r^3} \tag{9}$$

where r is the distance of point M from the charge e.

Since **H** is a solenoidal vector, that is, div $\mathbf{H}=0$, there exists for it a vector potential **b** such that $\mathbf{H}=\operatorname{curl}\mathbf{b}$ or, taking into account formula (9),

$$\operatorname{curl}\mathbf{b}=\frac{[e\mathbf{v},\ \mathbf{r}]}{4\pi r^3}=\frac{e}{4\pi}\frac{[\mathbf{v},\ \mathbf{r}]}{r^3}.$$

Let us rewrite this formula as

$$\operatorname{curl}\mathbf{b}=\frac{1}{4\pi}\left\{\left[e\mathbf{v},\ \frac{x}{r^3}\mathbf{i}\right]+\left[e\mathbf{v},\ \frac{y}{r^3}\mathbf{j}\right]+\left[e\mathbf{v},\ \frac{z}{r^3}\mathbf{k}\right]\right\}$$

$$=\frac{1}{4\pi}\left\{\left[\mathbf{i},\ -\frac{ex\mathbf{v}}{r^3}\right]+\left[\mathbf{j},\ -\frac{ey\mathbf{v}}{r^3}\right]+\left[\mathbf{k},\ -\frac{ez\mathbf{v}}{r^3}\right]\right\}$$

$$=\frac{1}{4\pi}\left\{\left[\mathbf{i},\ \frac{\partial}{\partial x}\left(\frac{e\mathbf{v}}{r}\right)\right]+\left[\mathbf{j},\ \frac{\partial}{\partial y}\left(\frac{e\mathbf{v}}{r}\right)\right]+\left[\mathbf{k},\ \frac{\partial}{\partial z}\left(\frac{e\mathbf{v}}{r}\right)\right]\right\}.$$

Employing the readily verifiable equation

$$\operatorname{curl}\mathbf{a}=\left[\mathbf{i},\ \frac{\partial\mathbf{a}}{\partial x}\right]+\left[\mathbf{j},\ \frac{\partial\mathbf{a}}{\partial y}\right]+\left[\mathbf{k},\ \frac{\partial\mathbf{a}}{\partial z}\right],$$

we obtain

$$\operatorname{curl}\mathbf{b}=\frac{1}{4\pi}\operatorname{curl}\frac{e\mathbf{v}}{r},$$

whence

$$\mathbf{b}=\frac{1}{4\pi}\cdot\frac{e\mathbf{v}}{r}.$$

Using formula (8), find the vector potentials of the following solenoidal fields defined in star-shaped regions:

262. $\mathbf{a}=\mathbf{i}$.

263. $\mathbf{a}=6x\mathbf{i}-15y\mathbf{j}+9z\mathbf{k}$.

264. $\mathbf{a}=5x^2y\mathbf{i}-10xyz\mathbf{k}$.

265. $\mathbf{a}=2\cos xz\cdot\mathbf{j}$.

266. $\mathbf{a}=\dfrac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2},\ x^2+y^2>0$.

CHAPTER VI

CURVILINEAR COORDINATES. BASIC OPERATIONS OF VECTOR ANALYSIS IN CURVILINEAR COORDINATES

Sec. 23. Curvilinear coordinates

In many problems it is more convenient to define the position of a point M in space by three numbers (q_1, q_2, q_3) instead of the three Cartesian coordinates (x, y, z). These numbers often prove to be more suitable to the problem at hand.

Let every point M be associated with a definite number triple (q_1, q_2, q_3) and, conversely, let every number triple correspond to a unique point M. Then the quantities q_1, q_2, q_3 are termed the *curvilinear coordinates* of the point M.

The *coordinate surfaces* in a system of curvilinear coordinates q_1, q_2, q_3 are the surfaces

$$q_1 = C_1, \qquad (1)$$
$$q_2 = C_2, \qquad (2)$$
$$q_3 = C_3, \qquad (3)$$

on which one of the coordinates remains constant.

The line of intersection of two coordinate surfaces is called a *coordinate line* (*axis*).

The coordinates q_2 and q_3 maintain constant values along the line of intersection of the coordinate surfaces (2) and (3); it is only the coordinate q_1 that varies. Similarly, on the lines of intersection of the surfaces (1) and (3) and (1) and (2), it is q_2 and q_3 that vary respectively.

Let us introduce the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ directed along tangents to the coordinate axes (q_1), (q_2), (q_3) at the point M in the direction of increasing variables q_1, q_2,

q_3 respectively (Fig. 37). Let us agree to take the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ always in that order so that, taken together, they constitute a right-handed trihedral.

The basic difference between curvilinear coordinates and Cartesian coordinates is this. In the Cartesian system, the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are constant for all points of space and are equal, respectively, to $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. In any other

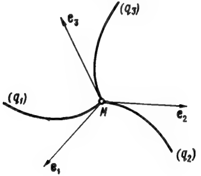

Fig. 37

system, they will, generally speaking, change their directions when passing from one point, M, to other points.

As examples of curvilinear coordinates we consider cylindrical and spherical coordinates.

1°. *Cylindrical coordinates*. The position of a point M in space is defined in cylindrical coordinates by three coordinates:

$$\begin{aligned} q_1 &= \rho, \quad 0 \leqslant \rho < +\infty, \\ q_2 &= \varphi, \quad 0 \leqslant \varphi < 2\pi, \\ q_3 &= z, \quad -\infty < z < +\infty. \end{aligned} \tag{4}$$

The coordinate surfaces are:

$\rho =$ constant: circular cylinders with the z-axis;

$\varphi =$ constant: half-planes adjoining the z-axis;

$z =$ constant: planes perpendicular to the z-axis.

The coordinate lines (or axes) are:

(ρ): rays perpendicular to the z-axis and having their origin on that axis;

(φ): circles with centre on the z-axis and lying in planes perpendicular to that axis;

(z): straight lines parallel to the z-axis (Fig. 38).

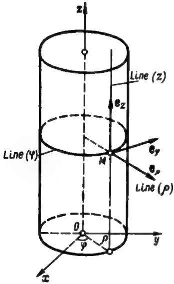

Fig. 38

Cartesian coordinates are related to cylindrical coordinates by the following formulas:

$$x = \rho \cos \varphi,$$
$$y = \rho \sin \varphi, \qquad (5)$$
$$z = z.$$

2°. *Spherical coordinates.* In spherical coordinates, the position of a point M in space is defined by the following coordinates:

$$q_1 = r, \quad 0 \leqslant r < +\infty,$$
$$q_2 = \theta, \quad 0 \leqslant \theta \leqslant \pi, \qquad (6)$$
$$q_3 = \varphi, \quad 0 \leqslant \varphi < 2\pi.$$

The coordinate surfaces are (Fig. 39):

r = constant: spheres centred at O:

θ = constant: circular half-angle cones with the z-axis;

φ = constant: half-planes adjoining the z-axis.

The coordinate lines are:

(r): rays emanating from the point O;

(θ): meridians on a sphere;

(φ): parallels on a sphere.

Cartesian coordinates are related to spherical coordinates via the following formulas:

$$x = r \cos \varphi \sin \theta,$$
$$y = r \sin \varphi \sin \theta, \qquad (7)$$
$$z = r \cos \theta.$$

A system of curvilinear coordinates is said to be *orthogonal* if at every point M the unit vectors e_1, e_2, e_3

are pairwise orthogonal. In such a system, the coordinate lines and the coordinate surfaces are also orthogonal. Systems of cylindrical and spherical coordinates are instances of orthogonal curvilinear systems of coordinates.

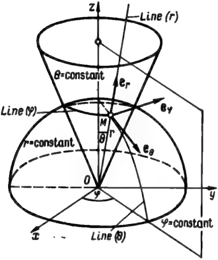

Fig. 39

Henceforth we consider only orthogonal systems of coordinates.

Suppose $\mathbf{r} = \mathbf{r}(q_1, q_2, q_3)$ is the radius vector of a point M. Then

$$d\mathbf{r} = H_1\, dq_1\, \mathbf{e}_1 + H_2\, dq_2\, \mathbf{e}_2 + H_3\, dq_3\, \mathbf{e}_3. \tag{8}$$

Here

$$H_i = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2}, \quad i = 1,\ 2,\ 3$$

are the Lamé coefficients of the given curvilinear system of coordinates.

In cylindrical coordinates,

$$q_1 = \rho, \quad q_2 = \varphi, \quad q_3 = z.$$

By virtue of (5) we have

$$H_1 = H_\rho = \sqrt{\left(\frac{\partial x}{\partial \rho}\right)^2 + \left(\frac{\partial y}{\partial \rho}\right)^2 + \left(\frac{\partial z}{\partial \rho}\right)^2} = 1,$$

$$H_2 = H_\varphi = \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2} = \rho,$$

$$H_3 = H_z = \sqrt{\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2} = 1.$$

In spherical coordinates,

$$q_1 = r,\ q_2 = \theta,\ q_3 = \varphi.$$

By virtue of (7) we have

$$H_1 = H_r = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = 1,$$

$$H_2 = H_\theta = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = r,$$

$$H_3 = H_\varphi = \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2} = r\sin\theta.$$

The quantities

$$dl_i = H_i\, dq_i, \quad i = 1,\ 2,\ 3$$

that appear in formula (8) are differentials of the arc lengths of the coordinates lines. In a number of cases, this consideration permits of a more simple computation of the Lamé coefficients. For instance, in the case of cylindrical coordinates (4) (see Fig. 38), the differentials of the arc lengths of the coordinates lines (ρ), (φ), (z) are

$$d(\rho) = 1 \cdot d\rho, \quad \text{whence } H_1 = 1;$$
$$d(\varphi) = \rho \cdot d\varphi, \quad \text{whence } H_2 = \rho;$$
$$d(z) = 1 \cdot dz, \quad \text{whence } H_3 = 1.$$

It is just as easy to obtain expressions for the Lamé coefficients in the case of spherical coordinates (6).

Sec. 24. Basic operations of vector analysis in curvilinear coordinates

1°. *Differential equations of vector lines.* Suppose we have a vector field

$$\mathbf{a} = a_1(q_1, q_2, q_3)\,\mathbf{e}_1 + a_2(q_1, q_2, q_3)\,\mathbf{e}_2 + a_3(q_1, q_2, q_3)\,\mathbf{e}_3.$$

The vector-line equations in curvilinear coordinates q_1, q_2, q_3 are of the form

$$\frac{H_1\,dq_1}{a_1(q_1, q_2, q_3)}=\frac{H_2\,dq_2}{a_2(q_1, q_2, q_3)}=\frac{H_3\,dq_3}{a_3(q_1, q_2, q_3)}.$$

In particular, in cylindrical coordinates ($q_1 = \rho$, $q_2 = \varphi$, $q_3 = z$):

$$\frac{d\rho}{a_1(\rho, \varphi, z)}=\frac{\rho\,d\varphi}{a_2(\rho, \varphi, z)}=\frac{dz}{a_3(\rho, \varphi, z)}; \tag{1}$$

in spherical coordinates ($q_1 = r$, $q_2 = \theta$, $q_3 = \varphi$):

$$\frac{dr}{a_1(r, \theta, \varphi)}=\frac{r\,d\theta}{a_2(r, \theta, \varphi)}=\frac{r\sin\theta\,d\varphi}{a_3(r, \theta, \varphi)}.$$

Example 1. A vector field is given in cylindrical coordinates

$$\mathbf{a}(M) = \mathbf{e}_\rho + \varphi\mathbf{e}_\varphi.$$

Find the vector lines of the field.

Solution. It is given that $a_1 = 1$, $a_2 = \varphi$, $a_3 = 0$. By virtue of formula (1) we have

$$\frac{d\rho}{1}=\frac{\rho\,d\varphi}{\varphi}=\frac{dz}{0},$$

whence

$$z = C_1,$$
$$\rho = C_2\varphi,$$

which are Archimedean spirals lying in planes parallel to the xy-plane.

2°. *The gradient in orthogonal coordinates.* Suppose we have a scalar field

$$u = u(q_1, q_2, q_3).$$

Then

$$\operatorname{grad} u=\frac{1}{H_1}\frac{\partial u}{\partial q_1}\mathbf{e}_1+\frac{1}{H_2}\frac{\partial u}{\partial q_2}\mathbf{e}_2+\frac{1}{H_3}\frac{\partial u}{\partial q_3}\mathbf{e}_3.$$

In particular, in cylindrical coordinates ($q_1 = \rho$, $q_2 = \varphi$, $q_3 = z$):

$$\operatorname{grad} u=\frac{\partial u}{\partial \rho}\mathbf{e}_\rho+\frac{1}{\rho}\frac{\partial u}{\partial \varphi}\mathbf{e}_\varphi+\frac{\partial u}{\partial z}\mathbf{e}_z; \tag{2}$$

in spherical coordinates ($q_1 = r$, $q_2 = \theta$, $q_3 = \varphi$):

$$\operatorname{grad} u=\frac{\partial u}{\partial r}\mathbf{e}_r+\frac{1}{r}\frac{\partial u}{\partial \theta}\mathbf{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi}\mathbf{e}_\varphi. \tag{3}$$

Example 2. Compute the gradient of the following scalar field specified in cylindrical coordinates (ρ, φ, z):

$$u = \rho + z \cos \varphi.$$

Solution. Using formula (2), we obtain

$$\operatorname{grad} u = 1 \cdot \mathbf{e}_\rho - \frac{1}{\rho} z \sin \varphi \cdot \mathbf{e}_\varphi + \cos \varphi \cdot \mathbf{e}_z.$$

Example 3. Find the gradient of the following scalar field given in spherical coordinates (r, θ, φ):

$$u = r + \frac{\sin \theta}{r} - \sin \theta \cos \varphi.$$

Solution. Using formula (3), we have

$$\operatorname{grad} u = \left(1 - \frac{\sin \theta}{r}\right) \mathbf{e}_r + \frac{\cos \theta}{r} \left(\frac{1}{r} - \cos \varphi\right) \mathbf{e}_\theta + \frac{\sin \varphi}{r} \mathbf{e}_\varphi.$$

3°. *The curl in orthogonal coordinates.* Suppose

$$\mathbf{a} = a_1 (q_1, q_2, q_3)\, \mathbf{e}_1 + a_2 (q_1, q_2, q_3)\, \mathbf{e}_2 + a_3 (q_1, q_2, q_3)\, \mathbf{e}_3.$$

Then

$$\operatorname{curl} \mathbf{a} = \begin{vmatrix} \frac{1}{H_2 H_3} \mathbf{e}_1 & \frac{1}{H_1 H_3} \mathbf{e}_2 & \frac{1}{H_1 H_2} \mathbf{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ a_1 H_1 & a_2 H_2 & a_3 H_3 \end{vmatrix}.$$

In particular, in cylindrical coordinates $(q_1 = \rho,\ q_2 = \varphi,\ q_3 = z)$:

$$\operatorname{curl} \mathbf{a} = \begin{vmatrix} \frac{1}{\rho} \mathbf{e}_\rho & \mathbf{e}_\varphi & \frac{1}{\rho} \mathbf{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ a_1 & \rho a_2 & a_3 \end{vmatrix}; \qquad (4)$$

in spherical coordinates $(q_1 = r,\ q_2 = \theta,\ q_3 = \varphi)$:

$$\operatorname{curl} \mathbf{a} = \begin{vmatrix} \frac{1}{r^2 \sin \theta} \mathbf{e}_r & \frac{1}{r \sin \theta} \mathbf{e}_\theta & \frac{1}{r} \mathbf{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ a_1 & r a_2 & r \sin \theta \cdot a_3 \end{vmatrix}. \qquad (4')$$

Example 4. Compute the curl of the following vector field specified in cylindrical coordinates:

$$\mathbf{a} = \sin\varphi \cdot \mathbf{e}_\rho + \frac{\cos\varphi}{\rho}\mathbf{e}_\varphi - \rho z \mathbf{e}_z.$$

Solution. Taking advantage of formula (4), we get

$$\operatorname{curl}\mathbf{a} = \begin{vmatrix} \frac{1}{\rho}\mathbf{e}_\rho & \mathbf{e}_\varphi & \frac{1}{\rho}\mathbf{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ \sin\varphi & \cos\varphi & -\rho z \end{vmatrix} = \frac{1}{\rho}\mathbf{e}_\rho(0-0) - \mathbf{e}_\varphi(-z-0)$$

$$+\frac{1}{\rho}\mathbf{e}_z(0-\cos\varphi) = z\mathbf{e}_\varphi - \frac{\cos\varphi}{\rho}\mathbf{e}_z.$$

4°. *The divergence in orthogonal coordinates.* Suppose we have a vector field

$$\mathbf{a} = a_1(q_1, q_2, q_3)\mathbf{e}_1 + a_2(q_1, q_2, q_3)\mathbf{e}_2 + a_3(q_1, q_2, q_3)\mathbf{e}_3.$$

Then

$$\operatorname{div}\mathbf{a} = \frac{1}{H_1H_2H_3}\left[\frac{\partial(a_1H_2H_3)}{\partial q_1} + \frac{\partial(a_2H_1H_3)}{\partial q_2} + \frac{\partial(a_3H_1H_2)}{\partial q_3}\right].$$

In particular, in cylindrical coordinates ($q_1 = \rho$, $q_2 = \varphi$, $q_3 = z$):

$$\operatorname{div}\mathbf{a} = \frac{1}{\rho}\frac{\partial(\rho a_1)}{\partial\rho} + \frac{1}{\rho}\frac{\partial a_2}{\partial\varphi} + \frac{\partial a_3}{\partial z};$$

in spherical coordinates ($q_1 = r$, $q_2 = \theta$, $q_3 = \varphi$):

$$\operatorname{div}\mathbf{a} = \frac{1}{r^2}\frac{\partial(r^2a_1)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\cdot a_2)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial a_3}{\partial\varphi}. \quad (5)$$

Example 5. Show that the vector field

$$\mathbf{a} = \frac{2\cos\theta}{r^3}\mathbf{e}_r + \frac{\sin\theta}{r^3}\mathbf{e}_\theta$$

is solenoidal.

Solution. Using formula (5), we have

$$\operatorname{div}\mathbf{a} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{2\cos\theta}{r^3}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\sin\theta}{r^3}\right) + 0$$

$$= \frac{1}{r^2}\left(-\frac{2\cos\theta}{r^2}\right) + \frac{1}{r^4\sin\theta}2\sin\theta\cos\theta = 0$$

wherever $r \neq 0$, which means that the field of the vector $\mathbf{a}$ is solenoidal at all points with the exception of $r = 0$.

267. Find the equations of the following vector fields:

(a) $\mathbf{a}=\mathbf{e}_\rho+\frac{1}{r}\mathbf{e}_\varphi+\mathbf{e}_z$;

(b) $\mathbf{a}=\rho\mathbf{e}_\rho+\varphi\mathbf{e}_\varphi+z\mathbf{e}_z$;

(c) $\mathbf{a}=\frac{2\alpha\cos\theta}{r^3}\mathbf{e}_r+\frac{\alpha\sin\theta}{r^3}\mathbf{e}_\theta$, $\alpha=\text{constant}$.

Find the gradients of the following scalar fields.

(a) In cylindrical coordinates:

268. $u = \rho^2 + 2\rho\cos\varphi - e^z\sin\varphi$.

269. $u = \rho\cos\varphi + z\sin^2\varphi - 3^\rho$.

(b) In spherical coordinates:

270. $u = r^2\cos\theta$.

271. $u = 3r^2\sin\theta + e^r\cos\varphi - r$.

272. $u=\mu\frac{\cos\theta}{r^2}$, $\mu=\text{constant}$.

Compute the divergence of the following vectors.

(a) In cylindrical coordinates:

273. $\mathbf{a} = \rho\mathbf{e}_\rho + z\sin\varphi\cdot\mathbf{e}_\varphi + e^\varphi\cos z\cdot\mathbf{e}_z$.

274. $\mathbf{a} = \varphi\arctan\rho\cdot\mathbf{e}_\rho + 2\mathbf{e}_\varphi - z^2e^z\mathbf{e}_z$.

(b) In spherical coordinates:

275. $\mathbf{a}=r^2\mathbf{e}_r-2\cos^2\varphi\cdot\mathbf{e}_\theta+\frac{\varphi}{r^2+1}\mathbf{e}_\varphi$.

Compute the curl of the following vector fields:

276. $\mathbf{a} = (2r + \alpha\cos\varphi)\,\mathbf{e}_r - \alpha\sin\theta\cdot\mathbf{e}_\theta + r\cos\theta\cdot\mathbf{e}_\varphi$, $\alpha = \text{constant}$.

277. $\mathbf{a} = r^2\mathbf{e}_r + 2\cos\theta\cdot\mathbf{e}_\theta - \varphi\mathbf{e}_\varphi$.

278. $\mathbf{a}=\cos\varphi\cdot\mathbf{e}_\rho-\frac{\sin\varphi}{\rho}\mathbf{e}_\varphi+\rho^2\mathbf{e}_z$.

279. Show that the vector field

$$\mathbf{a}=\frac{2\cos\theta}{r^3}\mathbf{e}_r+\frac{\sin\theta}{r^3}\mathbf{e}_\theta$$

is a potential field.

280. Show that the vector field

$$\mathbf{a} = f(r)\,\mathbf{e}_r,$$

where f is any differentiable function, is a potential field.

5°. *Computing the flux in curvilinear coordinates.* Let S be a part of the coordinate surface $q_1 = C$, where $C =$

$=$ constant, bounded by the coordinate lines

$$q_1 = \alpha_1, \quad q_2 = \alpha_2 \quad (\alpha_1 < \alpha_2);$$
$$q_3 = \beta_1, \quad q_3 = \beta_2 \quad (\beta_1 < \beta_2).$$

Then the flux of the vector

$$\mathbf{a} = a_1(q_1, q_2, q_3)\,\mathbf{e}_1 + a_2(q_1, q_2, q_3)\,\mathbf{e}_2 + a_3(q_1, q_2, q_3)\,\mathbf{e}_3$$

through the surface S in the direction of the vector $\mathbf{e}_1$ is computed from the formula

$$\Pi = \int_{\alpha_1}^{\alpha_2}\int_{\beta_1}^{\beta_2} a_1(C, q_2, q_3)\,H_2(C, q_2, q_3)\,H_3(C, q_2, q_3)\,dq_3\,dq_2. \tag{6}$$

The calculation is similar for the flux of a vector through a part of the surface $q_2 = C$ or through a part of the surface $q_3 = C$, where $C =$ constant.

Example 6. Compute the flux of the vector field, specified in cylindrical coordinates,

$$\mathbf{a} = \rho\mathbf{e}_\rho + z\mathbf{e}_\varphi$$

through the outer part of the lateral surface of the cylinder $\rho = 1$, bounded by the planes $z = 0$ and $z = 1$.

Solution. The cylinder is the coordinate surface $\rho = C =$ constant and so the desired flux

$$\Pi = \int_0^{2\pi}\int_0^1 C^2\,dz\,d\varphi = 2\pi C^2,$$

whence for the surface $\rho = 1$ we obtain

$$\Pi = 2\pi.$$

Example 7. Find the flux of the vector field, specified in spherical coordinates,

$$\mathbf{a} = r^2\theta\mathbf{e}_r + re^2\,\theta\mathbf{e}_\theta$$

through the outer side of an upper hemisphere S of radius R with centre at the coordinate origin.

Solution. The hemisphere S is part of the coordinate surface $r =$ constant, namely, $r = R$. On the surface S we have

$$q_1 = r = R; \quad q_2 = \theta, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2};$$
$$q_3 = \varphi, \quad 0 \leqslant \varphi < 2\pi.$$

Taking into account that in spherical coordinates

$$H_1 = H_r = 1, \quad H_2 = H_\theta = r, \quad H_3 = H_\varphi = r \sin\theta,$$

we find, via (6), that

$$\Pi = \int_0^{\pi/2} d\theta \int_0^{2\pi} R^4 \theta \sin\theta \, d\varphi = 2\pi R^4 \int_0^{\pi/2} \theta \sin\theta \, d\theta = 2\pi R^4.$$

Compute the flux of the vector field, specified in cylindrical coordinates, through the given surface S.

281. $\mathbf{a} = \rho \mathbf{e}_\rho - \cos\varphi \cdot \mathbf{e}_\varphi + z \mathbf{e}_z$; S is a closed surface formed by the cylinder $\rho = 2$ and by the planes $z = 0$ and $z = 2$.

282. $\mathbf{a} = \rho \mathbf{e}_\rho + \rho\varphi \mathbf{e}_\varphi - 2z \mathbf{e}_z$; S is a closed surface formed by the cylinder $\rho = 1$, the half-planes $\varphi = 0$ and $\varphi = \pi/2$, and by the planes $z = -1$ and $z = 1$.

283. Find the flux of the vector field $\mathbf{a} = (1/r^2)\, \mathbf{e}_r$ through a sphere of radius R with centre at the coordinate origin.

284. Find the flux of the vector, specified in spherical coordinates,

$$\mathbf{a} = r\mathbf{e}_r + r \sin\theta \cdot \mathbf{e}_\theta - 3r\varphi \sin\theta \cdot \mathbf{e}_\varphi$$

through an upper hemisphere of radius R.

285. Find the flux of the vector, specified in spherical coordinates,

$$\mathbf{a} = r^2 \mathbf{e}_r + R^2 \cos\varphi \cdot \mathbf{e}_\varphi$$

through the sphere $r = R$.

286. Find the flux of the vector, specified in spherical coordinates,

$$\mathbf{a} = r\mathbf{e}_r - r \sin\theta \cdot \mathbf{e}_\varphi$$

through a semicircle of radius R located in the half-plane $\varphi = \pi/4$ (the flux is taken in the direction of the vector $\mathbf{e}_\varphi$).

287. Find the flux of the vector, specified in spherical coordinates,

$$\mathbf{a} = r \sin\frac{\varphi}{2}\, \mathbf{e}_\theta + r \sin\theta \cos\varphi \cdot \mathbf{e}_\varphi$$

through the outer side of part of the half-angle cone $\sqrt{3z^2} = x^2 + y^2$, bounded from above by the plane $z = \sqrt{3}$ $(0 \leqslant z \leqslant \sqrt{3})$.

6°. *Finding the potential in curvilinear coordinates.* Given, in curvilinear coordinates q_1, q_2, q_3, the vector field

$$\mathbf{a}(M) = a_1(q_1, q_2, q_3)\,\mathbf{e}_1 + a_2(q_1, q_2, q_3)\,\mathbf{e}_2 + \\ + a_3(q_1, q_2, q_3)\,\mathbf{e}_3;$$

this is a potential field in some region Ω over which the variables q_1, q_2, q_3 range, that is, $\operatorname{curl}\mathbf{a} = 0$ in Ω.

To find the potential $u = u(q_1, q_2, q_3)$ of this field, write the equation $\mathbf{a}(M) = \operatorname{grad} u(M)$ as

$$a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = \frac{1}{H_1}\frac{\partial u}{\partial q_1}\mathbf{e}_1 + \frac{1}{H_2}\frac{\partial u}{\partial q_2}\mathbf{e}_2 + \frac{1}{H_3}\frac{\partial u}{\partial q_3}\mathbf{e}_3.$$

From this it follows that

$$\frac{\partial u}{\partial q_1} = a_1H_1, \quad \frac{\partial u}{\partial q_2} = a_2H_2, \quad \frac{\partial u}{\partial q_3} = a_3H_3. \tag{7}$$

This is a system of partial differential equations whose integration yields the desired potential $u = u(q_1, q_2, q_3) + C$, where C is an arbitrary constant.

The system (7) of differential equations is solved in the same way as in finding the potential in Cartesian coordinates.

The system (7) of differential equations is of the following form:

(1) In cylindrical coordinates ($q_1 = \rho$, $q_2 = \varphi$, $q_3 = z$),

$$\frac{\partial u}{\partial \rho} = a_\rho, \quad \frac{\partial u}{\partial \varphi} = \rho a_\varphi, \quad \frac{\partial u}{\partial z} = a_z. \tag{7'}$$

(2) In spherical coordinates ($q_1 = r$, $q_2 = \theta$, $q_3 = \varphi$),

$$\frac{\partial u}{\partial r} = a_r, \quad \frac{\partial u}{\partial \theta} = ra_\theta, \quad \frac{\partial u}{\partial \varphi} = r\sin\theta \cdot a_\varphi. \tag{7''}$$

Example 8. Find the potential of the following vector field specified in cylindrical coordinates:

$$\mathbf{a} = \left(\frac{\arctan z}{\rho} + \cos\varphi\right)\mathbf{e}_\rho - \sin\varphi\cdot\mathbf{e}_\varphi + \frac{\ln\rho}{1+z^2}\mathbf{e}_z.$$

Solution. By formula (4) we find

$$\operatorname{curl}\mathbf{a} = \begin{vmatrix} \frac{1}{\rho}\mathbf{e}_\rho & \mathbf{e}_\varphi & \frac{1}{\rho}\mathbf{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ \frac{\arctan z}{\rho} + \cos\varphi & -\rho\sin\varphi & \frac{\ln\rho}{1+z^2} \end{vmatrix} = 0 \quad (\rho > 0).$$

This is a potential field. The desired potential $u = u(\rho, \varphi, z)$ is the solution of the following system of differential equations:

$$\left.\begin{aligned} \frac{\partial u}{\partial \rho} &= \frac{\arctan z}{\rho} + \cos\varphi, \\ \frac{\partial u}{\partial \varphi} &= -\rho \sin\varphi, \\ \frac{\partial u}{\partial z} &= \frac{\ln \rho}{1+z^2}. \end{aligned}\right\}$$

From the first equation, integrating with respect to ρ we obtain

$$u = \ln \rho \cdot \arctan z + \rho \cos \varphi + C(\varphi, z). \qquad (8)$$

Differentiating (8) with respect to φ, we get

$$\frac{\partial u}{\partial \varphi} = -\rho \sin \varphi + \frac{\partial C}{\partial \varphi}$$

and since $\partial u/\partial \varphi = -\rho \sin \varphi$, it follows that $\partial C/\partial \varphi \equiv 0$, that is, $C = C_1(z)$. Thus

$$u = \ln \rho \cdot \arctan z + \rho \cos \varphi + C_1(z),$$

whence

$$\frac{\partial u}{\partial z} = \frac{\ln \rho}{1+z^2} + C_1'(z).$$

By virtue of the third equation of the system we have

$$\frac{\ln \rho}{1+z^2} = \frac{\ln \rho}{1+z^2} + C_1'(z)$$

or $C_1'(z) \equiv 0$, whence $C_1(z) \equiv C = \text{constant}$.

To summarize, the potential of the given field is

$$u(\rho, \varphi, z) = \ln \rho \cdot \arctan z + \rho \cos \varphi + C.$$

In the following problems, verify that the vector fields given in cylindrical coordinates are potential fields and find their potentials.

288. $\mathbf{a} = \mathbf{e}_\rho + \frac{1}{\rho}\mathbf{e}_\varphi + \mathbf{e}_z$.

289. $\mathbf{a} = \rho\mathbf{e}_\rho + \frac{\varphi}{\rho}\mathbf{e}_\varphi + z\mathbf{e}_z$.

290. $\mathbf{a} = \varphi z\mathbf{e}_\rho + z\mathbf{e}_\varphi + \rho\varphi\mathbf{e}_z$.

291. $\mathbf{a} = e^\rho \sin \varphi \cdot \mathbf{e}_\rho + \frac{1}{\rho} e^\rho \cos \varphi \cdot \mathbf{e}_\varphi + 2z\mathbf{e}_z$.

292. $\mathbf{a} = \varphi \cos z \cdot \mathbf{e}_\rho + \cos z \cdot \mathbf{e}_\varphi + \rho\varphi \sin z \cdot \mathbf{e}_z$.

Example 9. Find the potential of the following vector field given in spherical coordinates:

$$\mathbf{a} = \frac{1}{r} e^{\theta\varphi} \mathbf{e}_r + \frac{\theta \ln r}{r \sin\theta} e^{\theta\varphi} \mathbf{e}_\varphi + \frac{\ln r}{r} \varphi e^{\theta\varphi} \mathbf{e}_\theta.$$

Solution. Using (4'), we find that

$$\operatorname{curl} \mathbf{a} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta \cdot \mathbf{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ \frac{1}{r} e^{\theta\varphi} & \varphi \ln r \cdot e^{\theta\varphi} & \theta \ln r \cdot e^{\theta\varphi} \end{vmatrix} = 0.$$

This is a potential field in the region where $r > 0$, $\theta \neq n\pi$ $(n = 0, \pm 1, \ldots)$.

The system (7) of differential equations for finding the potential $u = u(r, \theta, \varphi)$ is of the form

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r} e^{\theta\varphi}, \\ \frac{\partial u}{\partial \theta} &= \varphi e^{\theta\varphi} \ln r, \\ \frac{\partial u}{\partial \varphi} &= \theta e^{\theta\varphi} \ln r. \end{aligned}\right\} \tag{9}$$

Integrating the first equation of system (9), we obtain

$$u = e^{\theta\varphi} \ln r + C(\varphi, \theta). \tag{10}$$

Differentiating (10) with respect to θ and taking into account the second equation of the system, we have

$$\varphi e^{\theta\varphi} \ln r = \varphi e^{\theta\varphi} \ln r + \frac{\partial C}{\partial \theta}.$$

That is, $\partial C/\partial\theta \equiv 0$, whence $C(\varphi, \theta) \equiv C_1(\varphi)$ and therefore

$$u = e^{\theta\varphi} \ln r + C_1(\varphi). \tag{11}$$

Differentiating (11) with respect to φ and taking into account the third equation of system (9), we obtain

$$\theta e^{\theta\varphi} \ln r = \theta e^{\theta\varphi} \ln r + C_1'(\varphi)$$

or $C_1'(\varphi) \equiv 0$, whence $C_1(\varphi) \equiv C = \text{constant}$. The desired potential is

$$u(r, \theta, \varphi) = e^{\theta\varphi} \ln r + C.$$

Establish the potentiality of the following vector fields, specified in spherical coordinates, and find their potentials.

293. $\mathbf{a} = \theta \mathbf{e}_r + \mathbf{e}_\theta$.

294. $\mathbf{a} = 2r\mathbf{e}_r + \frac{1}{r\sin\theta}\mathbf{e}_\varphi + \frac{1}{r}\mathbf{e}_\theta$.

295. $\mathbf{a} = \frac{1}{2}\varphi^2\mathbf{e}_r + \frac{\varphi}{\sin\theta}\mathbf{e}_\varphi + \frac{\theta}{r}\mathbf{e}_\theta$.

296. $\mathbf{a} = \cos\varphi\sin\theta\cdot\mathbf{e}_r + \cos\varphi\cos\theta\cdot\mathbf{e}_\theta - \sin\varphi\cdot\mathbf{e}_\varphi$.

297. $\mathbf{a} = e^r\sin\theta\cdot\mathbf{e}_r + \frac{1}{r}e^r\cos\theta\cdot\mathbf{e}_\theta + \frac{2\varphi}{(1+\varphi^2)\,r\sin\theta}\mathbf{e}_\varphi$.

7°. *Computing the line integral and the circulation of a vector field in curvilinear coordinates.* Suppose a vector field

$$\mathbf{a}(M) = a_1(q_1, q_2, q_3)\,\mathbf{e}_1 + a_2(q_1, q_2, q_3)\,\mathbf{e}_2 + a_3(q_1, q_2, q_3)\,\mathbf{e}_3$$

is defined and is continuous in a region Ω over which the orthogonal curvilinear coordinates q_1, q_2, q_3 range.

As we know [see Sec. 23, (8)], the differential $d\mathbf{r}$ of the radius vector $\mathbf{r}$ of any point $M(q_1, q_2, q_3) \in \Omega$ is equal to

$$d\mathbf{r} = H_1\,dq_1\,\mathbf{e}_1 + H_2\,dq_2\,\mathbf{e}_2 + H_3\,dq_3\,\mathbf{e}_3.$$

Therefore the line integral of the vector $\mathbf{a}(M)$ over an oriented smooth or piecewise smooth curve $L \subset \Omega$ is

$$\int_L (\mathbf{a}, d\mathbf{r}) = \int_L a_1H_1\,dq_1 + a_2H_2\,dq_2 + a_3H_3\,dq_3. \tag{12}$$

In particular, for the cylindrical coordinates $q_1 = \rho$, $q_2 = \varphi$, $q_3 = z$ we have

$$\mathbf{a} = a_\rho(\rho, \varphi, z)\,\mathbf{e}_\rho + a_\varphi(\rho, \varphi, z)\,\mathbf{e}_\varphi + a_z(\rho, \varphi, z)\,\mathbf{e}_z,$$
$$d\mathbf{r} = d\rho\cdot\mathbf{e}_\rho + \rho\,d\varphi\cdot\mathbf{e}_\varphi + dz\cdot\mathbf{e}_z,$$

and therefore

$$\int_L (\mathbf{a}, d\mathbf{r}) = \int_L a_\rho\,d\rho + a_\varphi\rho\,d\varphi + a_z\,dz. \tag{13}$$

For the spherical coordinates $q_1 = r$, $q_2 = \theta$, $q_3 = \varphi$ we have

$$\mathbf{a} = a_r (r, \theta, \varphi)\, \mathbf{e}_r + a_\theta (r, \theta, \varphi)\, \mathbf{e}_\theta + a_\varphi (r, \theta, \varphi)\, \mathbf{e}_\varphi,$$
$$d\mathbf{r} = dr \cdot \mathbf{e}_r + r\, d\theta \cdot \mathbf{e}_\theta + r \sin\theta\, d\varphi \cdot \mathbf{e}_\varphi,$$

and consequently

$$\int_L (\mathbf{a}, d\mathbf{r}) = \int_L a_r\, dr + r a_\theta\, d\theta + r a_\varphi \sin\theta\, d\varphi. \qquad (14)$$

The circulation C of the vector field $\mathbf{a}\,(M)$ is computed in the curvilinear coordinates q_1, q_2, q_3 via formula (12) in the general case; for cylindrical or spherical coordinates it is computed from (13) and (14) respectively.

Example 10. Compute the line integral in the vector field, given in cylindrical coordinates,

$$\mathbf{a} = 4\rho \sin\varphi \cdot \mathbf{e}_\rho + z e^\rho \mathbf{e}_\varphi + (\rho + \varphi)\, \mathbf{e}_z$$

along the straight line

$$L: \begin{cases} \varphi = \dfrac{\pi}{4}; \\ z = 0, \end{cases}$$

from the point $O\,(0, \pi/4, 0)$ to the point $A\,(1, \pi/4, 0)$.

Solution. In the given example,

$$a_\rho = 4\rho \sin\varphi, \quad a_\varphi = z e^\rho, \quad a_z = \rho + \varphi.$$

By formula (13), the desired line integral is

$$\int_L (\mathbf{a}, d\mathbf{r}) = \int_L 4\rho \sin\varphi\, d\rho + \rho z e^\rho\, d\varphi + (\rho + \varphi)\, dz.$$

On the straight line L we have

$$\varphi = \frac{\pi}{4}, \quad d\varphi = 0; \quad z = 0, \quad dz = 0; \quad 0 \leqslant \rho \leqslant 1.$$

Therefore

$$\int_L (\mathbf{a}, d\mathbf{r}) = \int_L 2\sqrt{2}\, \rho\, d\rho = \sqrt{2} \int_0^1 2\rho\, d\rho = \sqrt{2}.$$

Example 11. Compute the line integral in the vector field, given in spherical coordinates,

$$\mathbf{a} = e^r \sin\theta \cdot \mathbf{e}_r + 3\theta^2 \sin\varphi \cdot \mathbf{e}_\theta + r\varphi\theta \mathbf{e}_\varphi$$

along the line

$$L: \begin{cases} r=1, \\ \varphi=\frac{\pi}{2}, \end{cases} \quad 0 \leqslant \theta \leqslant \frac{\pi}{2},$$

in the direction from the point $M_0(1, 0, \pi/2)$ to the point $M_1(1, \pi/2, \pi/2)$ (Fig. 40).

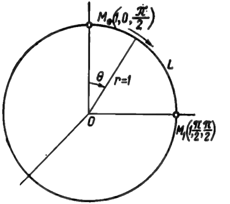

Fig. 40

Solution. The curve L is an arc of a circle with centre at the coordinate origin and radius $r = 1$ located in the yz-plane. The coordinates of this vector are

$$a_r = e^r \sin\theta, \quad a_\theta = 3\theta^2 \sin\varphi, \quad a_\varphi = r\varphi\theta.$$

By virtue of (14) the line integral is of the form

$$\int_L (\mathbf{a}, d\mathbf{r}) = \int_L e^r \sin\theta\, dr + 3\theta^2 r \sin\varphi\, d\theta + r^2\varphi\theta \sin\theta\, d\varphi.$$

Taking into account that the following conditions hold on L,

$$r=1, \ dr=0; \quad \varphi=\frac{\pi}{2}, \ d\varphi=0; \quad 0 \leqslant \theta \leqslant \frac{\pi}{2},$$

we obtain

$$\int_L (\mathbf{a}, d\mathbf{r}) = \int_L 3\theta^2 \, d\theta = \int_0^{\pi/2} 3\theta^2 \, d\theta = \frac{\pi^3}{8}.$$

Example 12. Compute the circulation of the vector field, given in cylindrical coordinates,

$$\mathbf{a} = \rho \sin \varphi \cdot \mathbf{e}_\rho + \rho z \mathbf{e}_\varphi + \rho^3 \mathbf{e}_z$$

over the curve

$$L: \begin{cases} \rho = \sin \varphi, \\ z = 0, \end{cases} \quad 0 \leqslant \varphi \leqslant \pi,$$

directly and via the Stokes theorem.

Solution. The coordinates of the vector are

$$a_\rho = \rho \sin \varphi, \quad a_\varphi = \rho z, \quad a_z = \rho^3.$$

The contour L is a closed curve located in the plane $z = 0$ (Fig. 41).

(1) Direct calculation of the circulation.

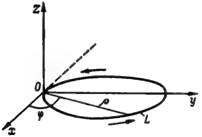

Fig. 41

Substituting the coordinates of the vector into (13), we obtain

$$C = \oint_L \rho \sin \varphi \, d\rho + \rho^2 z \, d\varphi + \rho^3 \, dz.$$

On the curve L we have

$$z = 0, \quad dz = 0; \quad \rho = \sin \varphi, \quad d\rho = \cos \varphi \, d\varphi,$$
$$0 \leqslant \varphi \leqslant \pi.$$

Therefore the desired circulation is equal to

$$C=\oint_L \rho \sin\varphi \, d\rho=\int_0^{\pi} \sin^2\varphi \cos\varphi \, d\varphi=0.$$

(2) Computing the circulation via the Stokes theorem.

By the Stokes theorem, the desired circulation is equal to

$$C=\oint_L (\mathbf{a},\, d\mathbf{r})=\iint_S (\operatorname{curl}\mathbf{a},\, \mathbf{n}^0)\, dS,$$

where S is the surface spanning the contour L.

We find the curl of the given field:

$$\operatorname{curl}\mathbf{a}=\frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\varphi & \mathbf{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ \rho\sin\varphi & \rho^2 z & \rho^3 \end{vmatrix}=-\rho\mathbf{e}_\rho-3\rho^2\mathbf{e}_\varphi + (2z-\cos\varphi)\,\mathbf{e}_z.$$

At points where $\rho = 0$ we redefine the value of curl $\mathbf{a}$ with respect to continuity, setting

$$\operatorname{curl}\mathbf{a}\,(0,\ \varphi,\ z) = (2z - \cos\varphi)\,\mathbf{e}_z.$$

Thus, curl $\mathbf{a}$ is defined throughout three-dimensional space. Since the curve L lies in the plane $z = 0$, for the surface S spanning this curve we take that portion of the plane $z = 0$ that is bounded by the curve L. Then we can take the unit vector $\mathbf{e}_z$ for the unit vector of the normal $\mathbf{n}^0$ to the surface S, that is, $\mathbf{n}^0 = \mathbf{e}_z$. We find the scalar product:

$$(\operatorname{curl}\mathbf{a},\ \mathbf{n}^0) = (-\rho\mathbf{e}_\rho - 3\rho^2\mathbf{e}_\varphi + (2z - \cos\varphi)\,\mathbf{e}_z,\ \mathbf{e}_z) = 2z - \cos\varphi$$

because by virtue of the orthonormality of the basis $\mathbf{e}_\rho$, $\mathbf{e}_\varphi$, $\mathbf{e}_z$ we have

$$(\mathbf{e}_\rho,\ \mathbf{e}_z) = (\mathbf{e}_\varphi,\ \mathbf{e}_z) = 0, \quad (\mathbf{e}_z,\ \mathbf{e}_z) = 1.$$

The desired circulation is

$$C=\iint_S (2z-\cos\varphi)\, dS.$$

Taking into account that $z=0$ on S and the element of area dS of the coordinate surface $z=0$ is equal to

$$dS = \rho\, d\rho\, d\varphi,$$

we finally get

$$C = -\iint_S \cos\varphi\, dS = -\iint_S \cos\varphi\rho\, d\rho\, d\varphi$$

$$= -\int_0^\pi \cos\varphi\, d\varphi \int_0^{\sin\varphi} \rho\, d\rho = -\frac{1}{2}\int_0^\pi \sin^2\varphi\cos\varphi\, d\varphi = 0.$$

Example 13. Compute the circulation of the vector, specified in spherical coordinates,

$$\mathbf{a} = r\mathbf{e}_r + (R+r)\sin\theta\cdot\mathbf{e}_\varphi$$

around the circle

$$L:\ \begin{cases} r = R \\ \theta = \frac{\pi}{2} \end{cases}$$

in the direction of increasing values of the angle φ, directly and via the Stokes theorem.

Solution. In this example,

$$a_r = r,\quad a_\theta = 0,\quad a_\varphi = (R+r)\sin\theta.$$

(1) Direct computation of the circulation.

By formula (14) the desired circulation is equal to

$$C = \oint_L r\,dr + (R+r)\sin\theta\, r\sin\theta\, d\varphi =$$

$$= \oint_L r\,dr + r(R+r)\sin^2\theta\, d\varphi.$$

On the given circle L, the centre of which lies at the coordinate origin, we have

$$r = R,\quad dr = 0;\quad \theta = \frac{\pi}{2};\quad 0 \leqslant \varphi < 2\pi,$$

and, consequently,

$$C = 2R^2\oint_L d\varphi = 2R^2\int_0^{2\pi} d\varphi = 4\pi R^2.$$

(2) Computation of the circulation via the Stokes theorem.

The desired circulation is, by the Stokes theorem, equal to

$$C = \oint_L (\mathbf{a},\ d\mathbf{r}) = \iint_S (\operatorname{curl} \mathbf{a},\ \mathbf{n}^0)\, dS,$$

where S is the surface spanning the circle L.

We find the curl of the given vector:

$$\operatorname{curl} \mathbf{a} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta \cdot \mathbf{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ r & 0 & (Rr + r^2)\sin^2\theta \end{vmatrix}$$

$$= \frac{2}{r}(R + r)\cos\theta \cdot \mathbf{e}_r - \frac{1}{r}(R + 2r)\sin\theta \cdot \mathbf{e}_\theta .$$

For the surface S spanning the circle L we take, for example, the upper hemisphere of radius R: $r = R$, $0 \leqslant \theta \leqslant \pi/2$, $0 \leqslant \varphi < 2\pi$. The unit vector of the normal $\mathbf{n}^0$ to the outer side of the hemisphere S is directed along the vector $\mathbf{e}_r$, and so we take $\mathbf{n}^0 = \mathbf{e}_r$. We find the scalar product

$$(\operatorname{curl} \mathbf{a},\ \mathbf{n}^0) = \left(\frac{2(R+r)}{r}\cos\theta \cdot \mathbf{e}_r - \frac{R+2r}{r}\sin\theta \cdot \mathbf{e}_\theta,\ \mathbf{e}_r\right)$$

$$= \frac{2(R+r)}{r}\cos\theta$$

since $(\mathbf{e}_r,\ \mathbf{e}_r) = 1$, $(\mathbf{e}_r,\ \mathbf{e}_\theta) = 0$.

Taking into account that $r = R$ on the surface S, we obtain the following expression for the desired flux:

$$\Pi = \iint_S \frac{2(R+r)}{r}\cos\theta\, dS = 4 \iint_S \cos\theta\, dS.$$

In spherical coordinates, the element of area dS of the coordinate surface $r = R$, that is, the hemisphere S, is equal to

$$dS = R^2 \sin\theta\, d\theta\, d\varphi$$

and, consequently,

$$\Pi = 4\iint_S \cos\theta\, R^2 \sin\theta\, d\theta\, d\varphi$$

$$= 4R^2 \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta \int_0^{2\pi} d\varphi = 4\pi R^2.$$

By the Stokes theorem we get $C = 4\pi R^2$.

For the surface S spanning the circle L we could take the lower hemisphere, the unit vector of the normal of which, $\mathbf{n}^0 = -\mathbf{e}_r$, and the result would be the same: $C = 4\pi R^2$.

Note that it is not desirable to take the circle bounded by L for the surface S spanning the circle L because the circle has a point $r = 0$ (the centre of the circle) in which the curl of the given vector has a discontinuity.

Compute the line integral along the given curves L in the following vector fields specified in cylindrical coordinates.

298. $\mathbf{a} = z\mathbf{e}_\rho + \rho\varphi\mathbf{e}_\varphi + \cos\varphi\cdot\mathbf{e}_z$; L is a segment of the straight line: $\{\rho = a,\ \varphi = 0,\ 0 \leqslant z \leqslant 1\}$.

299. $\mathbf{a} = \rho\mathbf{e}_\rho + 2\rho\varphi\mathbf{e}_\varphi + z\mathbf{e}_z$; L is the semicircle: $\{\rho = 1,\ z = 0,\ 0 \leqslant \varphi \leqslant \pi\}$.

300. $\mathbf{a} = e^\rho \cos\varphi\cdot\mathbf{e}_\rho + \rho\sin\varphi\cdot\mathbf{e}_\varphi + \rho\mathbf{e}_z$; L is a turn of the helical curve: $\{\rho = R,\ z = \varphi,\ 0 \leqslant \varphi \leqslant 2\pi\}$.

Compute the line integral over the given curve L in the following vector fields given in spherical coordinates.

301. $\mathbf{a} = e^r \cos\theta\cdot\mathbf{e}_r + 2\theta\cos\varphi\cdot\mathbf{e}_\theta + \varphi\mathbf{e}_\varphi$; L is the semicircle: $\{r = 1,\ \varphi = 0,\ 0 \leqslant \theta \leqslant \pi\}$.

302. $\mathbf{a} = 4r^3 \tan\frac{\varphi}{2}\,\mathbf{e}_r + \theta\varphi\mathbf{e}_\theta + \cos^2\varphi\cdot\mathbf{e}_\varphi$; L is a segment of the straight line: $\left\{\varphi = \frac{\pi}{2};\ \theta = \frac{\pi}{4},\ 0 \leqslant r \leqslant 1\right\}$.

303. $\mathbf{a} = \sin^2\theta\cdot\mathbf{e}_r + \sin\theta\cdot\mathbf{e}_\theta + r\varphi\theta\mathbf{e}_\varphi$; L is a segment of the straight line: $\left\{\varphi = \frac{\pi}{2},\ r = \frac{1}{\sin\theta},\ \frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{2}\right\}$.

Compute the circulation of the following vector fields, specified in cylindrical coordinates, about the given con-

tours, directly and with the aid of the Stokes theorem.

304. $\mathbf{a} = z\mathbf{e}_\rho + \rho z\mathbf{e}_\varphi + \rho\mathbf{e}_z$; L is the circle: $\{\rho = 1, z = 0\}$.

305. $\mathbf{a} = \rho \sin\varphi\cdot\mathbf{e}_\rho - \rho^2 z\mathbf{e}_\varphi + \rho^2\mathbf{e}_z$; L is the circle: $\{\rho = R, z = R\}$.

306. $\mathbf{a} = z\cos\varphi\cdot\mathbf{e}_\rho + \rho\mathbf{e}_\varphi + \varphi^2\mathbf{e}_z$; L is the loop: $\{\rho = \sin\varphi, z = 1\}$.

Compute the circulation of the following vectors, given in spherical coordinates, along the given contours L, directly and with the aid of the Stokes theorem.

307. $\mathbf{a} = r\theta\mathbf{e}_r + r\sin\theta\cdot\mathbf{e}_\varphi$; L is the circle: $\left\{r = 1, \theta = \frac{\pi}{4}\right\}$.

308. $\mathbf{a} = r\sin\theta\cdot\mathbf{e}_r + \theta e^\theta\mathbf{e}_\theta$; L is the loop: $\left\{r = \sin\varphi, \theta = \frac{\pi}{2}, 0 \leqslant \varphi \leqslant \pi\right\}$.

309. $\mathbf{a} = r\varphi\theta\mathbf{e}_\varphi$; L is a contour bounded by the semicircle: $\left\{r = R, \varphi = \frac{\pi}{4}, 0 \leqslant \theta \leqslant \pi\right\}$ and its vertical diameter $\left\{\varphi = \frac{\pi}{4}, \theta = 0\right\}$.

Sec. 25. The Laplace operator in orthogonal coordinates

If $u = u(q_1, q_2, q_3)$ is a scalar function, then

$$\operatorname{grad} u = \frac{1}{H_1}\frac{\partial u}{\partial q_1}\mathbf{e}_1 + \frac{1}{H_2}\frac{\partial u}{\partial q_2}\mathbf{e}_2 + \frac{1}{H_3}\frac{\partial u}{\partial q_3}\mathbf{e}_3. \qquad (1)$$

If

$$\mathbf{a} = a_1(q_1, q_2, q_3)\mathbf{e}_1 + a_2(q_1, q_2, q_3)\mathbf{e}_2 + a_3(q_1, q_2, q_3)\mathbf{e}_3,$$

then

$$\operatorname{div} \mathbf{a} = \frac{1}{H_1H_2H_3}\left[\frac{\partial}{\partial q_1}(a_1H_2H_3) + \frac{\partial}{\partial q_2}(a_2H_3H_1) + \frac{\partial}{\partial q_3}(a_3H_1H_2)\right]. \qquad (2)$$

Using formulas (1) and (2), we obtain the following expression for the Laplace operator Δu:

$$\Delta u = \operatorname{div} \operatorname{grad} u = \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} \left(\frac{H_2 H_3}{H_1} \frac{\partial u}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{H_1 H_3}{H_2} \frac{\partial u}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{H_1 H_2}{H_3} \frac{\partial u}{\partial q_3} \right) \right].$$

In cylindrical coordinates,

$$\Delta u = \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{\partial}{\partial \varphi} \left(\frac{1}{\rho} \frac{\partial u}{\partial \varphi} \right) + \frac{\partial}{\partial z} \left(\rho \frac{\partial u}{\partial z} \right) \right] = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2}.$$

In spherical coordinates,

$$\Delta u = \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r} \left(r^2 \sin\theta \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial u}{\partial \theta} \right) + \frac{\partial}{\partial \varphi} \left(\frac{1}{\sin\theta} \frac{\partial u}{\partial \varphi} \right) \right] = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 u}{\partial \varphi^2}.$$

Example. Find all the solutions of the Laplace equation $\Delta u = 0$ that depend solely on the distance r.

Solution. Writing the Laplace equation in spherical coordinates and taking into account the spherical symmetry of the solution (it must not depend on θ or φ), we have

$$\Delta u \equiv \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) = 0 \qquad (u = u(r)),$$

whence

$$r^2 \frac{\partial u}{\partial r} = C_1$$

so that

$$u = \frac{C_1}{r} + C_2,$$

where C_1 and C_2 are constants.

310. Given: a scalar field $u = u(M)$ in cylindrical coordinates

$$u(\rho, \varphi, z) = \rho^2 \varphi + z^2 \varphi^3 - \rho\varphi z.$$

Find Δu.

311. Given: a scalar field $u = u(M)$ in spherical coordinates

$$u(r, \theta, \varphi) = r^2\varphi\theta + r^3\varphi^2 + \varphi + \theta^2.$$

Find Δu.

312. Are the following functions harmonic?

(1) $u = \rho^2 \cos 2\varphi$.

(2) $u = r \cos 2\theta$.

313. Find all possible harmonic functions that

(a) depend on θ alone,

(b) depend on φ alone

(in the spherical system of coordinates).

314. Find all solutions of the Poisson equation

$$\Delta u = r^{n-1}$$

in the spherical system of coordinates, provided $u = u(r)$.

ANSWERS

1. (a) The half-line $\begin{cases} x=2, \\ y=-z,\ y\geqslant 0,\ z\leqslant 0 \end{cases}$ is traversed twice when $-\infty < t < +\infty$. (b) When $t \in (-\infty, -1) \cup (-1, +\infty)$ the point $\mathbf{r}(t) = \frac{t^2+1}{(t+1)^2}\mathbf{i} + \frac{2t}{(t+1)^2}\mathbf{j}$ twice traverses the half-line $x+y=1$, $x\geqslant\frac{1}{2}$, $y\leqslant\frac{1}{2}$.

(c) $\left.\begin{matrix} x^2+y^2=1, \\ z=1; \end{matrix}\right\}$. (d) $y=\frac{x^2}{3}$, $z=\frac{x^3}{9}$.

(e) $x^2+y^2+z^2=1$, $x-y=0$.

7. $\mathbf{i}+\mathbf{k}$. 8. $\mathbf{i}+\mathbf{k}$. 9. $-\mathbf{j}+\frac{1}{2\pi}\mathbf{k}$.

10. $-\mathbf{i}+\mathbf{k}$. 11. $e\mathbf{i}-\mathbf{j}+2\mathbf{k}$. 12. No. 14. No.

17. (a) $2\left(\frac{d\mathbf{r}}{dt}\mathbf{r}\right)$; (b) $\left|\frac{d\mathbf{r}}{\partial t}\right|^2+\left(\mathbf{r}, \frac{d^2\mathbf{r}}{dt^2}\right)$; (c) $\left[\mathbf{r}, \frac{d^2\mathbf{r}}{dt^2}\right]$.

21. Circles lying in planes perpendicular to the vector $\mathbf{a}$.
22. The hodograph of velocity is a helical curve: $x = a\cos t$, $y = a\sin t$, $z = 2bt$; the hodograph of acceleration is a circle: $x = -a\sin t$, $y = a\cos t$, $z = 2b$.

26. $\frac{d\mathbf{a}}{dt}=\frac{d\mathbf{a}}{du}\frac{du}{dt}$; $\frac{d^2\mathbf{a}}{dt^2}=\frac{d^2\mathbf{a}}{du^2}\left(\frac{du}{dt}\right)^2+\frac{d\mathbf{a}}{du}\frac{d^2u}{dt^2}$.

28. $(t-1)e^t\mathbf{i}+\frac{1}{2}\left(t-\frac{1}{2}\sin 2t\right)\mathbf{j}-\arctan t\cdot\mathbf{k}+\mathbf{c}$.

29. $\frac{1}{2}\ln(1+t^2)\cdot\mathbf{i}+\frac{1}{2}e^{t^2}\cdot\mathbf{j}+\sin t\cdot\mathbf{k}+\mathbf{c}$.

30. $e^{\sin t}\cdot\mathbf{i}-\frac{1}{2}\sin t^2\cdot\mathbf{j}+t\mathbf{k}+\mathbf{c}$.

31. $\frac{t^3}{6}\mathbf{i}+(t\cos t-\sin t)\mathbf{j}+\frac{2^t}{\ln 2}\mathbf{k}+\mathbf{c}$.

32. $\frac{2}{3}\mathbf{j}+\pi\mathbf{k}$. 33. $(1-e^{-1/2})\,\mathbf{i}+(e^{1/2}-1)\,\mathbf{j}+(e-1)\,\mathbf{k}$.

34. $-\ln 2\cdot\mathbf{j}+\mathbf{k}$. 35. $2\pi^2\mathbf{i}+\pi\mathbf{j}+\pi^2\mathbf{k}$. 36. $R=\frac{\sqrt{2}}{|\sin 2t|}$.

37. $R=\frac{2}{3}|t|\,(1+9t^2)^{3/2}$. 38. $R=6$. 39. $R=\frac{1}{2}a\pi$.

40. $R=2a\cosh^2 t$. 41. $x+y=0$.

42. $x-y-\sqrt{2}z=0$. 43. $\frac{1}{T}=\frac{1}{3}$. 44. $\frac{1}{T}=\frac{1}{2a\cosh^2 t}$.

45. $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=C$, a family of triaxial ellipsoids.

46. $x^2+y^2-z=C$, a family of paraboloids.
47. $x^2+y^2=Cz$, a family of paraboloids.
48. $2y^2+9z^2=C$, a family of elliptical cylinders.
49. $x+2y-z=C$, a family of parallel planes.
50. A family of planes resulting from the sheaf of planes $a_1x+a_2y+a_3z=C\,(b_1x+b_2y+b_3z)$ passing through the straight line

$$\left.\begin{aligned} a_1x+a_2y+a_3z=0,\\ b_1x+b_2y+b_3z=0. \end{aligned}\right\}$$

via the elimination of the straight line itself. Here, a_1, a_2, a_3 are the coordinates of the vector **a**; b_1, b_2, b_3 are the coordinates of the vector **b**.
51. $x^2+y^2+z^2=C^2$, a family of concentric spheres.

52. $(\mathbf{a},\mathbf{b},\mathbf{r})=C$ or $\begin{vmatrix} x & y & z\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=C$, a family of parallel planes.

53. $2x-y=C$, a family of parallel straight lines.
54. $y=Cx$, $C>0$, $x\neq 0$, a family of rays.
55. $y^2=Cx$ is a family of parabolas with the vertex $O\,(0,\,0)$ deleted.
56. $x^2-y^2=C$, a family of hyperbolas.

57. $y = -x \ln C - C$, $C > 0$, a family of straight lines.

58. $\frac{\sqrt{15}}{5}$. 59. $\frac{-\sqrt{21}}{3}$. 60. $\frac{\sqrt{3}}{3}e^3$. 61. $-\frac{2}{5}$.

62. $\frac{3}{5}\sqrt{2}$. 63. $\frac{1}{4}$. 64. 0. 65. $\frac{2\sqrt{3}}{3}(\sqrt{2}+3)$. 66. 0.

67. -2. 68. $\frac{\pi a^2}{\sqrt{a^2+R^2}}$. 69. $\frac{2}{3}(\mathbf{i}+\mathbf{j}-\mathbf{k})$.

70. $\mathbf{k}$. 71. $\varphi=\pi$. 72. $\varphi=0$.

73. $\varphi = 0$. 74. $y = -x + 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$.

75. $x^2+y^2+z^2=1$. 78. $\frac{\mathbf{r}}{r^3}$. 79. $\mathbf{a}$. 80. $\mathbf{a}(\mathbf{b}, \mathbf{r})+\mathbf{b}(\mathbf{a}, \mathbf{r})$.

81. $2|\mathbf{a}|^2\mathbf{r}-2(\mathbf{a}, \mathbf{r})\mathbf{a}$. 86. $\frac{\partial u}{\partial r}=\frac{2u}{r}$.

87. $\frac{\partial u}{\partial l}=-\frac{\cos(\widehat{\mathbf{r}, \mathbf{l}})}{r^2}$; $\frac{\partial u}{\partial l}=0$ for $\mathbf{r}\perp\mathbf{l}$.

88. $\frac{\partial u}{\partial l}=\frac{1}{r^2}$. 89. $\frac{\partial u}{\partial l}=1$.

90. $\frac{\partial u}{\partial l}=\frac{(\operatorname{grad} u, \operatorname{grad} v)}{|\operatorname{grad} v|}$; $\frac{\partial u}{\partial l}=0$ if $\operatorname{grad} u \perp \operatorname{grad} v$.

91. (a) 1 in the direction of the y-axis; (b) 3 in the direction of the vector $\mathbf{a}=-\mathbf{i}-2\mathbf{j}+2\mathbf{k}$.

92. $y=\mathbf{e}_1 x$; $z=\mathbf{e}_2 x$. 93. $y=\frac{a_2}{a_1}x+C_1$; $z=\frac{a_3}{a_1}x+C_2$.

94. $\left.\begin{array}{l} x^2+y^2+z^2=C_1^2, \\ x+y+z=C_2. \end{array}\right\}$ 95. $\frac{1}{x}-\frac{1}{z}=1$, $\frac{1}{x}+\frac{1}{2y^2}=4$.

96. $x^2=C_1 y$, $z=C_2$. 97. $z=C_1 x$, $y=C_2$. 98. $xy=C_1$, $z=C_2$.

99. $x=C_1$, $2y^2-z^2=C_2$. 100. $\frac{1}{x}-\frac{1}{y}=C_1$, $z=C_2$.

101. $y^2+z^2=C_1$, $x=C_2$. 102. $x=C_1 y$, $x=C_2 z$.

103. $\left.\begin{array}{l} \frac{x}{b_{01}}=\frac{y}{b_{02}}+C_1, \\ \frac{x}{b_{01}}=\frac{z}{b_{03}}+C_3, \end{array}\right\}$

where b_{01}, b_{02}, b_{03} are cooordinates of the vector $\mathbf{b}_0$.
104. $\Pi = -3$. **105.** $\Pi = \pi R^2\gamma$. **106.** $\Pi = \pi R^2 h$.

107. $\Pi = 4\pi R^3 f(R)$. **108.** $\Pi = \frac{a^3}{2}$. **109.** $\Pi = \frac{\pi}{6}$.

110. $\Pi = \frac{1}{2}\pi R^2 h$. **111.** $\Pi = \pi h^3$.

112. $\Pi = \frac{81}{8}\pi$. **113.** $\Pi = \frac{\pi}{4}$.

114. $\Pi = 0$. **115.** $\Pi = \frac{\pi}{2}$. **116.** $\Pi = \frac{1}{4}$. **117.** $\Pi = 4\pi R^3$.

118. (a) $\Pi = -\frac{7}{6}$; (b) $\Pi = -\frac{1}{2}$; (c) $\Pi = \pi$. **119.** $\Pi = 0$.

120. $\Pi = 6\pi R$. **121.** $\Pi = 0$.

122. $\Pi = \pi$. **123.** $\Pi = 0$. **124.** $\Pi = \frac{2}{3}\pi\left(1 - \frac{\sqrt{2}}{4}\right)$.

125. $\Pi = \frac{3}{8}R^4$.

126. $\Pi = \sqrt{2}\,\pi$. **127.** $\Pi = 45\pi$. **128.** $\Pi = \frac{256}{3}\pi$.

129. $\Pi = 0$.

130. $\Pi = \pi$. **131.** $\psi(r) = \frac{C}{r}$. **132.** $7r^4$. **133.** 0. **134.** 0.

135. $\psi(z) = C - z$, C = constant. **136.** $\Pi = 4\pi R^3$.
137. $\operatorname{div} \mathbf{E} = 0$ $(r \neq 0)$.

143. 16π. **144.** πH^3. **145.** $\frac{32}{3}\pi$. **146.** 0. **147.** $\frac{\pi}{3}$.

148. 4π. **149.** $\frac{19}{3}\pi$. **150.** $\frac{32}{3}\pi$. **151.** $2R^3$.

152. $\frac{81}{8}\pi$. **153.** -1. **154.** $-\pi$.

155. Solenoidal field.
156. Nonsolenoidal field.
157. Solenoidal field.

159. $\varphi(r) = \frac{C}{r^3}$, $r \neq 0$, C = constant.

161. $\frac{r_2^2 - r_1^2}{2}$. **162.** $\ln \frac{r_2}{r_1}$. **163.** $\frac{1}{r_1} - \frac{1}{r_2}$. **164.** 0.

166. $-\frac{4}{3}R^2$. 167. $\frac{41}{6}$. 168. (a) $-\frac{14}{15}$; (b) $\frac{2}{3}$. 169. 0.

170. $\frac{5}{3}$. 171. $3\sqrt{3}$. 172. $\frac{1}{35}$.

173. $-\pi a^2$. 174. 1.

175. -2π. 176. $-\frac{\pi R^3}{4}$. 177. $\frac{4}{3}$.

179. $-2(z\mathbf{i}+x\mathbf{j}+y\mathbf{k})$.

180. $3(z^2-x^2)\mathbf{j}$. 181. $(x+y)\mathbf{k}$. 191. $\omega=\frac{1}{2}\operatorname{curl}\mathbf{v}=-\frac{z}{2}\mathbf{j}$.

193. $f(x, z)=xz+x+z+C$, $C=$ constant.

195. 4π. 196. -4π. 197. $\frac{4}{3}$. 198. -2π. 199. $\frac{128}{3}$.

200. 729π. 201. 0. 202. $-\sqrt{2}\,\pi$.

203. $2\omega\pi a^2$. Hint: $\mathbf{v}=[\boldsymbol{\omega}, \mathbf{r}]$.

204. $\mu_c=1$. 205. $\mu_c=3$. 206. Dependent.

207. Independent. 208. Dependent. 209. -1. 210. $\frac{4}{15}$.

211. 0. 212. $\frac{2}{3}$. 213. $\frac{\pi}{2}$. 214. $\frac{1}{3}$.

216. $\frac{\pi}{2}$. *Hint*: Supplement the path of integration L with the line segment OA of the x-axis.

217. No. 218. Yes. 219. No. 220. Yes. 221. No. 222. No. 223. Yes. 226. $\varphi = x^2yz$. 227. $\varphi = x + xyz$. 228. $\varphi = x^2y - y^2 + xz$. 229. $\varphi = \ln|x + y + z|$. 230. $\varphi = \arctan(xyz)$. 231. $\varphi = r$. 232. $\varphi = \ln r$.

233. $\varphi=\frac{1}{3}r^3$. 234. $\varphi=\alpha x+\beta y+\gamma z+C$, $C=$ constant.

235. $\varphi = xy + yz + zx + C$. 236. $\varphi = xy + e^z + C$.
237. $\varphi = e^x \sin y + z + C$. 247. (a) Yes, (b) No, (c) Yes.

249. $u = C_1x + C_2$.

250. $u = Ax^2 + Bxy + Ay^2$, where A and B are arbitrary.

251. $$u(x)=\begin{cases} \dfrac{x^n}{(n-1)n}+C_1x+C_2 & \text{if } n\neq 1, \\ x\ln|x|+C_1x+C_2 & \text{if } n=1\ (x\neq 0). \end{cases}$$

252. $I=\frac{-4}{15}\pi$. 253. $I=-\frac{\pi}{3}$. 254. $I=\frac{4}{3}\pi R^3$.
255. $I=0$.
256. $\mathbf{b}=x\mathbf{j}+(y-x)\mathbf{k}$. 257. $\mathbf{b}=(y^2-2xz)\mathbf{k}$.
258. $\mathbf{b}=(e^x-xe^y)\mathbf{j}$. 259. $\mathbf{b}=3x^2\mathbf{j}+(2y^3-6xz)\mathbf{k}$.
260. $\mathbf{b}=-x(x+y^2)\mathbf{j}+(x^3+y^3)\mathbf{k}$.
261. $\mathbf{b}=-(xz^2+yze^{x^2})\mathbf{j}-2xyz\mathbf{k}$.
262. $\mathbf{b}=\frac{1}{2}(-z\mathbf{j}+y\mathbf{k})$.
263. $\mathbf{b}=-8yz\mathbf{i}+xz\mathbf{j}+7xy\mathbf{k}$.
264. $\mathbf{b}=2xy^2z\mathbf{i}-3x^2yz\mathbf{j}+x^2y^2\mathbf{k}$.
265. $\mathbf{b}=\frac{1}{x}\sin xz\cdot\mathbf{i}-\frac{1}{z}\sin xz\cdot\mathbf{k}$.
266. $\mathbf{b}=\frac{x\mathbf{i}+y\mathbf{j}}{x^2+y^2}z-\mathbf{k}$. 267. (a) $\rho=\varphi+C_1$, $\rho=z+C_2$;
(b) $\rho=-\frac{1}{\ln C_1\varphi}$, $\rho=C_2z$; (c) $\varphi=C_1$, $r=C_2\sin^2\theta$.

268. $\operatorname{grad} u=2(\rho+\cos\varphi)\mathbf{e}_\rho-\left(2\sin\varphi+\frac{1}{\rho}e^z\cos\varphi\right)\mathbf{e}_\varphi - \mathbf{e}^z\sin\varphi\cdot\mathbf{e}_z$.

269. $\operatorname{grad} u=(\cos\varphi-3^\rho\ln 3)\mathbf{e}_\rho+\left(\frac{z}{\rho}\sin 2\varphi-\sin\varphi\right)\mathbf{e}_\varphi + \sin^2\varphi\cdot\mathbf{e}_z$.

270. $\operatorname{grad} u=2r\cos\theta\cdot\mathbf{e}_r-r\sin\theta\cdot\mathbf{e}_\theta$.

271. $\operatorname{grad} u=(6r\sin\theta+e^r\cos\varphi-1)\mathbf{e}_r+3r\cos\theta\cdot\mathbf{e}_\theta - \frac{e^r\sin\varphi}{r\sin\theta}\mathbf{e}_\varphi$.

272. $\operatorname{grad} u=-\mu\left(\frac{2\cos\theta}{r^3}\mathbf{e}_r+\frac{\sin\theta}{r^3}\mathbf{e}_\theta\right)$.

273. $\operatorname{div}\mathbf{a}=2+\frac{z}{\rho}\cos\varphi-e^\varphi\sin z$.

274. $\operatorname{div}\mathbf{a}=\frac{\varphi}{\rho}\arctan\rho+\frac{\varphi}{1+\rho^2}-(z^2+2z)e^z$.

275. $\operatorname{div}\mathbf{a}=4r-\frac{2}{r}\cos^2\varphi\cot\theta+\frac{1}{r(r^2+1)\sin\theta}$.

276. $\operatorname{curl}\mathbf{a}=\frac{\cos 2\theta}{\sin\theta}\mathbf{e}_r-\left(2\cos\theta+\frac{\alpha\sin\varphi}{r\sin\theta}\right)\mathbf{e}_\theta-\frac{\alpha\sin\theta}{r}\mathbf{e}_\varphi$.

277. curl $\mathbf{a} = -\frac{\varphi}{r}\cot\theta\cdot\mathbf{e}_r + \frac{\varphi}{r}\mathbf{e}_\theta + \frac{2\cos\theta}{r}\mathbf{e}_\varphi$.

278. curl $\mathbf{a} = -2\rho\mathbf{e}_\varphi + \frac{\sin\varphi}{\rho}\mathbf{e}_z$.

281. 24π. 282. $\frac{1}{2}\pi$. 283. 4π. 284. $\frac{2}{3}\pi R^3$.

285. $4\pi R^4$. 286. $-\frac{2}{3}R^3$.

287. 48. *Hint*: Write the equations of the surfaces in spherical coordinates.
288. $u = \rho + \varphi + z + C$.

289. $u = \frac{1}{2}(\rho^2 + \varphi^2 + z^2) + C$. 290. $u = \rho\varphi z + C$.

291. $u = e^\rho \sin\varphi + z^2 + C$. 292. $u = \rho\varphi\cos z + C$.
293. $u = r\theta + C$. 294. $u = r^2 + \varphi + \theta + C$.

295. $u = \frac{1}{2}(r\varphi^2 + \theta^2) + C$. 296. $u = r\cos\varphi\sin\theta + C$.

297. $u = e^r\sin\theta + \ln(1 + \varphi^2) + C$.
298. 1. 299. π^2. 300. $2\pi R$. 301. π^2. 302. 1.

303. $\frac{\pi}{4} + \frac{\sqrt{2}}{2} - 1$.

304. 0. 305. $-2\pi R^4$. 306. π.

307. π. 308. 0. 309. 0.

310. $\Delta u = 4\varphi - \frac{\varphi z}{\rho} + \frac{6\varphi z^2}{\rho^2} + 2\varphi^3$.

311. $\Delta u = 6\varphi\theta + 12r\varphi^2 + \frac{2}{r^2} + \varphi\cot\theta + \frac{2\theta}{r^2}\cot\theta + \frac{2r}{\sin^2\theta}$.

312. (1) Yes, (2) No.

313. (1) $u(\theta) = C_1\ln\left|\tan\frac{\theta}{2}\right| + C_2$, (2) $u(\varphi) = C_1\varphi + C_2$.

314. $$u(r) = \begin{cases} \frac{r^{n+1}}{(n+1)(n+2)} + \frac{C_1}{r} + C_2, & n \neq -1, -2, \\ \ln r + \frac{C_1}{r} + C_2, & n = -1, \ (r \neq 0) \\ -\frac{\ln r}{r} + \frac{C_1}{r} + C_2, & n = -2. \end{cases}$$

APPENDIX I

BASIC OPERATIONS OF VECTOR ANALYSIS IN ORTHOGONAL CURVILINEAR COORDINATES

1. The scalar field is given in orthogonal curvilinear coordinates, $u = u(q_1, q_2, q_3)$. Then we have

$$\operatorname{grad} u = \frac{1}{H_1}\frac{\partial u}{\partial q_1}\mathbf{e}_1 + \frac{1}{H_2}\frac{\partial u}{\partial q_2}\mathbf{e}_2 + \frac{1}{H_3}\frac{\partial u}{\partial q_3}\mathbf{e}_3.$$

The Laplace operator is

$$\Delta u = \frac{1}{H_1H_2H_3}\left[\frac{\partial}{\partial q_1}\left(\frac{H_2H_3}{H_1}\frac{\partial u}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{H_1H_3}{H_2}\frac{\partial u}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{H_1H_2}{H_3}\frac{\partial u}{\partial q_3}\right)\right].$$

Special cases: (a) The scalar field is given in cylindrical coordinates, $u = u(\rho, \varphi, z)$. Then we have

$$\operatorname{grad} u = \frac{\partial u}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial u}{\partial \varphi}\mathbf{e}_\varphi + \frac{\partial u}{\partial z}\mathbf{e}_z.$$

The Laplace operator is

$$\Delta u = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2}.$$

(b) The scalar field is given in spherical coordinates, $u = u(r, \theta, \varphi)$. Then we have

$$\operatorname{grad} u = \frac{\partial u}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi}\mathbf{e}_\varphi.$$

The Laplace operator is

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\sin\theta\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}.$$

2. The vector field is given in orthogonal curvilinear coordinates:

$$\mathbf{a} = a_1(q_1, q_2, q_3)\,\mathbf{e}_1 + a_2(q_1, q_2, q_3)\,\mathbf{e}_2 + a_3(q_1, q_2, q_3)\,\mathbf{e}_3.$$

Then we have

$$\operatorname{div} \mathbf{a} = \frac{1}{H_1H_2H_3}\left[\frac{\partial(a_1H_2H_3)}{\partial q_1} + \frac{\partial(a_2H_1H_3)}{\partial q_2} + \frac{\partial(a_3H_1H_2)}{\partial q_3}\right],$$

$$\operatorname{curl} \mathbf{a} = \begin{vmatrix} \frac{1}{H_2H_3}\mathbf{e}_1 & \frac{1}{H_1H_3}\mathbf{e}_2 & \frac{1}{H_1H_2}\mathbf{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ a_1H_1 & a_2H_2 & a_3H_3 \end{vmatrix}.$$

Special cases: (a) The vector field is given in cylindrical coordinates,

$$\mathbf{a} = a_1(\rho, \varphi, z)\,\mathbf{e}_\rho + a_2(\rho, \varphi, z)\,\mathbf{e}_\varphi + a_3(\rho, \varphi, z)\,\mathbf{e}_z.$$

Then we have

$$\operatorname{div} \mathbf{a} = \frac{1}{\rho}\frac{\partial(\rho a_1)}{\partial \rho} + \frac{1}{\rho}\frac{\partial a_2}{\partial \varphi} + \frac{\partial a_3}{\partial z},$$

$$\operatorname{curl} \mathbf{a} = \begin{vmatrix} \frac{1}{\rho}\mathbf{e}_\rho & \mathbf{e}_\varphi & \frac{1}{\rho}\mathbf{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ a_1 & \rho a_2 & a_3 \end{vmatrix}.$$

(b) The vector field is given in spherical coordinates:

$$\mathbf{a} = a_1(r, \theta, \varphi)\,\mathbf{e}_r + a_2(r, \theta, \varphi)\,\mathbf{e}_\theta + a_3(r, \theta, \varphi)\,\mathbf{e}_\varphi.$$

Then we have

$$\operatorname{div} \mathbf{a} = \frac{1}{r^2}\frac{\partial(a_1r^2)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(a_2\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial a_3}{\partial\varphi},$$

$$\operatorname{curl} \mathbf{a} = \begin{vmatrix} \frac{1}{r^2\sin\theta}\mathbf{e}_r & \frac{1}{r\sin\theta}\mathbf{e}_\theta & \frac{1}{r}\mathbf{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ a_1 & ra_2 & a_3r\sin\theta \end{vmatrix}.$$

APPENDIX II

AREA ELEMENTS OF COORDINATE SURFACES

Coordinates	Coordinate surfaces	Area elements
General q_1, q_2, q_3	$q_1 = C = \text{constant}$ $q_2 = C = \text{constant}$ $q_3 = C = \text{constant}$	$dS_1 = H_2(C, q_2, q_3)\, H_3(C, q_2, q_3)\, dq_2\, dq_3$ $dS_2 = H_1(q_1, C, q_3)\, H_3(q_1, C, q_3)\, dq_1\, dq_3$ $dS_3 = H_1(q_1, q_2, C)\, H_2(q_1, q_2, C)\, dq_1\, dq_2$
Cylindrical $q_1 = \rho$ $q_2 = \varphi$ $q_3 = z$	$\rho = C = \text{constant}$ $\varphi = C = \text{constant}$ $z = C = \text{constant}$	$dS = C\, d\varphi\, dz$ $dS = d\rho\, dz$ $dS = \rho\, d\rho\, d\varphi$
Spherical $q_1 = r$ $q_2 = \theta$ $q_3 = \varphi$	$r = C = \text{constant}$ $\theta = C = \text{constant}$ $\varphi = C = \text{constant}$	$dS = C^2 \sin\theta\, d\theta\, d\varphi$ $dS = r \sin C\, dr\, d\varphi$ $dS = r\, dr\, d\theta$

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This text is designed for students of engineering colleges and also for practising engineers who feel a need to refreshen their knowledge of such an important area of higher mathematics as vector analysis. Each section of the text starts out with a brief review of the essentials of theory (propositions, definitions, formulas). This is followed by detailed solutions of examples and problems.

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